Some results for a class of conformable fractional dynamic equations on time scales

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Abstract: In this paper, we discuss a class of fractional dynamic equations with damping term on time scales in the teaching of the college course Mathematical Physics Equations, and derive some results for it. The obtained results can be used in the research of oscillation properties for this kind of equations on time scales.

Key–Words: fractional dynamic equations; time scales; conformable fractional derivative; damping

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1 Introduction

Dynamic equations with damping term on time scales are important equations in the teaching of the college course Mathematical Physics Equations. Recently, there have many works related to this domain [1-10].

A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$.

Definition 1.1. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.

Definition 1.2. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f | f \in \mathfrak{R},\ 1 + \mu(t)f(t) > 0,\ \forall t \in \mathbb{T}\}$.

Definition 1.3: For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in \mathfrak{U}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 1.4: For $p \in \mathfrak{R}$, the *exponential function* is defined by

$$e_p(t, s) = \exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau) \text{ for } s, t \in \mathbb{T}.$$

If $\mathbb{T}=\mathbb{R}$, then

$$e_p(t,s)=\exp(\int_s^t p(\tau)d\tau), \quad for \quad s,\ t\in\mathbb{R},$$

If $\mathbb{T}=\mathbb{Z}$, then

$$e_p(t,s)=\prod_{\tau=s}^{t-1}[1+p(\tau)], \quad for \quad s,\ t\in\mathbb{Z} \quad and \quad s<t.$$

The following two theorems include some known properties on the *exponential function*.

Theorem 1.5 [11, Theorem 5.1]. If $p\in\Re$, and fix $t_0\in\mathbb{T}$, then the *exponential function* $e_p(t,t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^{\Delta}(t)=p(t)y(t), \\ y(t_0)=1. \end{cases}$$

Theorem 1.6 [11, Theorem 5.2]. If $p\in\Re^+$, then $e_p(t,s)>0$ for $\forall s,\ t\in\mathbb{T}$.

Recently, Benkhettou etc. developed a conformable fractional calculus theory on arbitrary time scales [12], and established the basic tools for fractional differentiation and fractional integration on time scales.

Definition 1.7 [12, Definition 1]. For $t\in\mathbb{T}^{\kappa}$, $\alpha\in(0,1]$, and a function $f\in(\mathbb{T},\mathbb{R})$, the fractional derivative of α order for f at t is denoted by $f^{(\alpha)}(t)$ (provided it exists) with the property such that for every $\varepsilon>0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$|[f(\sigma(t))-f(s)]t^{1-\alpha}-f^{(\alpha)}(t)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s| \text{ for all } s\in\mathfrak{U}.$$

Definition 1.8 [12, Definition 28]. If $F^{(\alpha)}(t)=f(t),\ t\in\mathbb{T}^{\kappa}$, then F is called an α-order *antiderivative* of f, and the Cauchy α-fractional integral of f is defined by

$$\int_a^b f(t)\Delta^{\alpha}t=\int_a^b f(t)t^{\alpha-1}\Delta t=F(b)-F(a), \text{ where } a,\ b\in\mathbb{T}.$$

Theorem 1.9 [12, Theorem 4]. For $t\in\mathbb{T}^{\kappa}$, $\alpha\in(0,1]$, and a function $f\in(\mathbb{T},\mathbb{R})$, the following conclusions hold:

(i). If f is conformal fractional differentiable of order α at $t>0$, then f is continuous at t.

(ii). If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with $f^{(\alpha)}(t)=\dfrac{f(\sigma(t))-f(t)}{\sigma(t)-t}t^{1-\alpha}=\dfrac{f(\sigma(t))-f(t)}{\mu(t)}t^{1-\alpha}$.

(iii). If t is right-dense, then f is conformable fractional differentiable of order α at t if, and only if, the limit $\lim\limits_{s\to t}\dfrac{f(s)-f(t)}{s-t}t^{1-\alpha}$ exists as a finite number. In this case, $f^{(\alpha)}(t)=\lim\limits_{s\to t}\dfrac{f(s)-f(t)}{s-t}t^{1-\alpha}$.

(iv). If f is fractional differentiable of order α at t, then $f(\sigma(t))=f(t)+\mu(t)t^{1-\alpha}f^{(\alpha)}(t)$.

Corollary 1.10. According to the definition of the conformable fractional differentiable of order α, it holds that $f^{(\alpha)}(t)=t^{1-\alpha}f^{\Delta}(t)$, where $f^{\Delta}(t)$ is the usual Δ derivative in the case $\alpha=1$. Furthermore, if $f^{(\alpha)}(t)>0\ (<0)$ for $t>0$, then f is increasing (decreasing) for $t>0$.

By a combination of Theorem 1.5 and Corollary 1.10 one can obtain the following theorem.

Theorem 1.11: Let $\widetilde{p}(t)=t^{\alpha-1}p(t)$, $\alpha\in(0,1]$. If $\widetilde{p}\in\Re$, and fix $t_0\in\mathbb{T}$, then the *exponential function* $e_{\widetilde{p}}(t,t_0)$ is the unique solution of the following initial value problem

$$\begin{cases} y^{(\alpha)}(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Theorem 1.12 [12, Theorem 15]. Assume $f,\ g \in (\mathbb{T}, \mathbb{R})$ are conformable fractional differentiable of order α. Then

(i). $(f+g)^{(\alpha)}(t) = f^{(\alpha)}(t) + g^{(\alpha)}(t)$.

(ii). $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(\sigma(t))g^{(\alpha)}(t) = f^{(\alpha)}(t)g(\sigma(t)) + f(t)g^{(\alpha)}(t)$.

(iii). $(\frac{1}{f})^{(\alpha)}(t) = -\frac{f^{(\alpha)}(t)}{f(t)f(\sigma(t))}$.

(iv). $(\frac{f}{g})^{(\alpha)}(t) = \frac{f^{(\alpha)}(t)g(t) - f(t)g^{(\alpha)}(t)}{g(t)g(\sigma(t))}$.

Motivated by the analysis above, in this paper, we will consider the following fractional dynamic equation with damping term on time scales of the following form:

$$(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} + p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)} + q(t)x(t) = 0,\ t \in \mathbb{T}_0, \tag{1.1}$$

where $\alpha \in (0,1]$, $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0 = [t_0, \infty) \bigcap \mathbb{T}$, $t_0 > 0$, $a,\ r,\ p,\ q \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

2 Main Results

Theorem 2.1. Suppose $-\frac{\widetilde{p}}{a} \in \mathfrak{R}_+$, and assume that

$$\int_{t_0}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s, t_0)}{a(s)} \Delta^\alpha s = \infty, \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \Delta^\alpha s = \infty, \tag{2.2}$$

and Eq. (1.1) has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$. Then we have the following statements:

(i). There exists a sufficiently large t_1 such that $(\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t, t_0)})^{(\alpha)} < 0,\ [r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_1, \infty)_{\mathbb{T}}$.

(ii). If furthermore assume that

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} [\frac{1}{r(\xi)} \int_{\xi}^{\infty} (\frac{e_{-\frac{\widetilde{p}}{a}}(\tau, t_0)}{a(\tau)} \int_{\tau}^{\infty} \frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s), t_0)} \Delta^\alpha s) \Delta^\alpha \tau] \Delta^\alpha \xi = \infty, \tag{2.3}$$

then either there exists a sufficiently large t_4 such that $x^{(\alpha)}(t) > 0$ on $[t_4, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$.

Proof of (i). By $-\frac{\widetilde{p}}{a} \in \mathfrak{R}_+$ and Theorem 1.6, we have $e_{-\frac{\widetilde{p}}{a}}(t, t_0) > 0$. Since x is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$, by Theorem 1.12 (iv) and Theorem 1.11 we obtain that

$$\begin{aligned}(\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t, t_0)})^{(\alpha)} &= \frac{e_{-\frac{\widetilde{p}}{a}}(t, t_0)(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} - (e_{-\frac{\widetilde{p}}{a}}(t, t_0))^{(\alpha)} a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t, t_0) e_{-\frac{\widetilde{p}}{a}}(\sigma(t), t_0)} \\ &= \frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} + p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t), t_0)} = \frac{-q(t)x(t)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t), t_0)} < 0. \end{aligned} \tag{2.4}$$

According to Corollary 1.10 one can see $\dfrac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)}$ is decreasing on $[t_0,\infty)_{\mathbb{T}}$. Furthermore, by Theorem 1.6 one has $e_{-\frac{\tilde{p}}{a}}(t,t_0) > 0$. So considering $a(t) > 0$ one can obtain that $[r(t)x^{(\alpha)}(t)]^{(\alpha)}$ is eventually of one sign. We claim $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Otherwise, assume there exists a sufficiently large $t_2 > t_1$ such that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} < 0$ on $t \in [t_2,\infty)_{\mathbb{T}}$. Then from Corollary 1.10 one can see $r(t)x^{(\alpha)}(t)$ is decreasing on $[t_2,\infty)_{\mathbb{T}}$, and from Definition 1.8 it holds that

$$r(t)x^{(\alpha)}(t) - r(t_2)x^{(\alpha)}(t_2) = \int_{t_2}^{t} \frac{e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{e_{-\frac{\tilde{p}}{a}}(s,t_0)a(s)}\Delta^\alpha s$$
$$\leq \frac{a(t_2)[r(t_2)x^{(\alpha)}(t_2)]^{(\alpha)}}{e_{-\frac{\tilde{p}}{a}}(t_2,t_0)} \int_{t_2}^{t} \frac{e_{-\frac{\tilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s. \tag{2.5}$$

It follows from (2.1) that $\lim_{t\to\infty} r(t)x^{(\alpha)}(t) = -\infty$, and thus there exists a sufficiently large $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $r(t)x^{(\alpha)}(t) < 0$ on $[t_3,\infty)_{\mathbb{T}}$. So

$$x(t) - x(t_3) = \int_{t_3}^{t} \frac{r(s)x^{(\alpha)}(s)}{r(s)}\Delta^\alpha s \leq r(t_3)x^{(\alpha)}(t_3)\int_{t_3}^{t} \frac{1}{r(s)}\Delta^\alpha s.$$

Due to (2.2) one can deduce that $\lim_{t\to\infty} x(t) = -\infty$, which leads to a contradiction. So it holds that $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_1,\infty)_{\mathbb{T}}$, and the proof is complete.

Proof of (ii). According to (i), since $[r(t)x^{(\alpha)}(t)]^{(\alpha)} > 0$ on $[t_1,\infty)_{\mathbb{T}}$, from Corollary 1.10 one can see that $x^{(\alpha)}(t)$ is eventually of one sign. So there exists a sufficiently large $t_4 > t_1$ such that either $x^{(\alpha)}(t) > 0$ or $x^{(\alpha)}(t) < 0$ on $[t_4,\infty)_{\mathbb{T}}$.

If $x^{(\alpha)}(t) < 0$, then $x(t)$ is decreasing, and considering $x(t)$ is a positive solution of Eq. (1.1) on $[t_0,\infty)_{\mathbb{T}}$, one can obtain that $\lim_{t\to\infty} x(t) = \beta_1 \geq 0$ and $\lim_{t\to\infty} r(t)x^{(\alpha)}(t) = \beta_2 \leq 0$. We claim $\beta_1 = 0$. Otherwise, assume $\beta_1 > 0$. Then there exists t_5 such that $x(t) \geq \beta_1$ on $[t_5,\infty)_{\mathbb{T}}$, and fulfilling α-fractional integral for (2.4) from t to ∞ yields

$$-\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)} = -\lim_{t\to\infty}\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\tilde{p}}{a}}(t,t_0)} + \int_t^\infty \frac{-q(s)x(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s$$
$$\leq -\int_t^\infty \frac{q(s)x(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s \leq -\beta_1\int_t^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s,$$

which is followed by

$$-[r(t)x^{(\alpha)}(t)]^{(\alpha)} \leq -\beta_1\Big[\frac{e_{-\frac{\tilde{p}}{a}}(t,t_0)}{a(t)}\int_t^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big]. \tag{2.6}$$

Substituting t with τ in (2.6), fulfilling α-fractional integral for (2.6) with respect to τ from t to ∞ yields

$$r(t)x^{(\alpha)}(t) = \lim_{t\to\infty} r(t)x^{(\alpha)}(t) - \beta_1\int_t^\infty\Big(\frac{e_{-\frac{\tilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau$$
$$= \beta_2 - \beta_1\int_t^\infty\Big(\frac{e_{-\frac{\tilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau$$
$$\leq -\beta_1\int_t^\infty\Big(\frac{e_{-\frac{\tilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty \frac{q(s)}{e_{-\frac{\tilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s\Big)\Delta^\alpha\tau,$$

which implies

$$x^{(\alpha)}(t) \leq -\beta_1[\frac{1}{r(t)}\int_t^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]. \tag{2.7}$$

Substituting t with ξ in (2.7), fulfilling α-fractional integral for (2.7) with respect to ξ from t_5 to t yields

$$x(t)-x(t_t) \leq -\beta_1\int_{t_5}^t[\frac{1}{r(\xi)}\int_\xi^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi. \tag{2.8}$$

By (2.8) and (2.3) we have $\lim_{t\to\infty}x(t)=-\infty$, which leads to a contradiction. So it holds that $\beta_1=0$. The proof is completed.

Theorem 2.2. Suppose $-\frac{\widetilde{p}}{a}\in\mathfrak{R}_+$, and assume that x is a positive solution of Eq. (1.1) such that

$$[r(t)x^{(\alpha)}(t)]^{(\alpha)}>0,\ x^{(\alpha)}(t)>0 \text{ on } [t_1,\infty)_{\mathbb{T}},$$

where $t_1\geq t_0$ is sufficiently large. Then we have

$$x^{(\alpha)}(t)\geq\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}].$$

Proof. From Theorem 2.1 one can see that $\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}$ is decreasing on $[t_1,\infty)$. So

$$r(t)x^{(\alpha)}(t)\geq r(t)x^{(\alpha)}(t)-r(t_1)x^{(\alpha)}(t_1)=\int_{t_1}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)[r(s)x^{(\alpha)}(s)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)a(s)}\Delta^\alpha s$$

$$\geq\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}\int_{t_1}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s=\delta_1(t,t_1)\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)},$$

and then

$$x^{(\alpha)}(t)\geq\frac{\delta_1(t,t_1)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}].$$

The proof is completed.

3 Conclusions

We have established some new results for a class of fractional dynamic equation with damping term on time scales. These results can be used in the research of oscillation properties for this kind of equations, which is supposed to further research.

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