Solving linear programming problems via weighted least-squares method

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Abstract. The Gaussian elimination method is usually used for solving problems related to linear programming. The paper describes an approximate method which solves a non-negative least-squares (NNLS) problem. The presented method is especially suitable for degenerate and unstable problems and also when a feasible initial solution is not known. The main ideas are explained by simple examples.

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1 Introduction

During the last 70 years, the simplex method, which is based on Gaussian elimination, has been used the most in solving linear programming (LP) and related problems. In some settings, however, it has performed poorly - and 30 years ago a foray of research in this area commenced. For degenerate and unstable problems the least-squares method is recommended. This highly developed method is much older than the simplex method, it is used not only in mathematics but also in statistics, physics, etc. Mainly nonlinear

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problems are solved by composing a certain number of similar linear least-squares problems, differing in a variable or constraint. In this paper it will be proved that such an idea can be used also for solving mathematical programming problems. First of all, the least-squares method is recommended for degenerate and unstable problems.

The basic problem used in this paper is the NNLS problem (non-negative least-squares), minimizing $\|Ex-b\|^2$ s.t.$x \geq 0$, see Section 2. This problem is equivalent to phase I algorithm for the simplex method discussed by Leichner, Dantzig and Davis (1993), see /1/. Their algorithm solves least-squares subproblems and guarantees strict improvement on degenerate problems at each step. A similar algorithm based on the least-squares method was described by Übi (2007,2010).

The least-squares method is described thoroughly in books by Björck (1996) and Lawson and Hanson (1995).

The main goal of this article is to show how the least-squares method can be used for solving degenerate and unstable linear programming problems.

2 Algorithm LS1 for the non-negative least squares (NNLS) problem

Let us have a non-negative least-squares (NNLS) problem, an overdetermined or underdetermined system of linear equations

$$Ex = b, x \geq 0, \tag{1}$$

or

$$min\{\varphi(x) = 0.5 \parallel Ex - b \parallel^2\}, x \geq 0,$$

where E is an $m \times n$ matrix, $x \in R^n, b \in R^m$, see Björck (1996), Lawson and Hanson (1995). In these books the main attention is paid to overdetermined systems, but in the mathematical programming $m < n$.

Assumption 2.1. *The columns of the matrix E and vector b are unit vectors,*

$$\|E_j\| = 1, j = 1, ..., n,$$
$$\|b\| = 1.$$

Our method for the problem (1) is described in paper Übi (2007, 2010). It corresponds to the first version of the second method, see Lawson-Hanson

(1995), ch 24. For the matrix E the QR transformation is computed, where Q is an orthogonal and R an upper triangular auxiliary matrix. The matrix E is not transformed. The order of matrix R is one at the first iteration, two at the second iteration etc. Choose the starting point $x^0 = 0$, the working set of columns is empty. At each step one variable x_{j0} is activated (for which column E_{j0} forms a minimal angle to the residual $\rho = b - Ex_E$ and column E_{j0} is added to the triangular matrix R) or one variable $x_j \leq 0$ (and its corresponding column in R) is removed. In the last case all the columns of R corresponding to this and following variables are replaced by the originals from the system (1). Then these Householder's transformations which were performed before the variable $x_j(x_j \leq 0)$ are applied to the replaced columns of R. At last the replaced part is transformed into triangular form keeping the reflection normals in the subdiagonal part of R. The Householder's transformations are memorized as products.

During the actual solution process the inequality $x_j > 0$ holds almost always. All variables x_j have been positive even in the case of having $m = 100$ constraints, when the elements of A, b and c were randomly generated. The use of algorithm LS1 for solving medium-sized problems has been documented by Übi /2007/, /2010/.

3 Solving linear programming problems

We consider the linear programming problem

$$z = (c, x) \rightarrow max \tag{2}$$

$$Ax = b$$

$$x \geq 0$$

and its dual

$$w = (y, b) \rightarrow min$$

$$yA \geq c, \tag{3}$$

where A is $m \times n$ matrix, b and y are $m-$vectors, x and c are $n-$vectors.

Often problem (2) does not have the feasible initial solution that is a prerequisite for using the simplex method and thus we minimize the sum of artificial variables during phase-I. The minimization of this sum has two setbacks - it is often the most computationally intensive part of the job and also the initial solution found does not depend on the objective function. On the other hand finding the initial solution is an NNLS problem. The goal of this paper is to find optimal solution $x*$ to the problem (2) by using only one NNLS problem.

If vector $\hat{x}$ is a feasible basic solution to the problem (2) and $\hat{z} = (c, \hat{x})$ the corresponding value of objective function, then we set $\hat{u} = \hat{x}/\hat{z}, \hat{t} = 1/\hat{z}$. We obtain a vector $\hat{u}$ and a scalar $\hat{t}$, a feasible solutions to the LP problem

$$\begin{aligned} z_1 = t \rightarrow min \\ (c, u) \quad = 1, \quad Au - tb = 0, \\ u, \, t \geq 0. \end{aligned} \tag{4}$$

Assumption 3.1. The right hand sides are non-negative, $b_i \geq 0 (i = 1, ..., m)$ and the maximum of the objective function satisfies the inequality $z_{max} > 1$.

In the case the assumption is not satisfied, we add any of the constraints to the objective function whilst possibly multiplying it with a positive number, see Remark 3.1. In the remark 3.5 we will further consider the case when the right hand side $b = 0$.

Thus in the case the assumption 3.1 is satisfied we may solve problem(4) instead of the initial problem (2). We will compose a following NNLS problem for finding an approximate solution to the problem (4)

$$\begin{aligned} (c, u) \quad = 1, \quad Au - tb = 0, \\ \epsilon t = 0, \\ u, \, t \geq 0, \end{aligned} \tag{5}$$

where $\epsilon > 0$ is a miniscule weighting coefficient.

If in the optimal solution to the problem (5) the optimal value of the variable t is strictly positive, $t* > 0$ and vector $u*/t*$ satisfies the constraints of the initial problem with a certain accuracy (the accuracy depends on the chosen coefficient ϵ) then we have obtained an approximate value of the optimal solution to the initial problem.

We will next consider the case when the minimum $t* = 0$.

Theorem of alternative 3.1. **(Gale,1969)** *The system $Dx \leq f$ has no solution if and only if there is a vector y such that*

$$(y, f) = -1, yD = 0, y \geq 0. \tag{6}$$

If the objective function of the initial problem is unbounded, then according to the theorem of duality the system $-yA \leq -c$ does not hold and due to the Gale's theorem there exists a nonnegative vector v, such that $Av = 0$ and $(c, v) = 1$. In this case the minimum $t* = 0$ and the least-squares problem (5) has such a solution, which satisfies all equations exactly, see Example 3.5.

Let us solve the problem (5). If the optimal value $t* = 0$, then the objective function of the initial problem (2) is unbounded or the problem is contradictory. If $t* > 0$ and vector $u * /t*$ does not satisfy the constraints of the initial problem within the chosen accuracy, then the problem is contradictory.

Remark 3.1. We have assumed that $z_{max} > 1$ while using the method described above. In a real-life problem is it usually possible to determine, whether the maximum of the objective function is positive or negative. As state above, we will add some constraints to the objective function, in case the maximum is negative. It may also be necessary to do the opposite, subtract some constraints from the objective function, in case the values of the objective function are too big, as in the problem (5) $t_{min} = 1/z_{max}$. It is possible to avoid excessively small values of t this way. We may also change the norm of the objective vector c.

Remark 3.2. The NNLS problem (5) my be solved using Matlab's built-in NNLS solver. In this paper the algorithm LS1 that is described in section 2, is deployed.

Remark 3.3. It is possible to prove that for $\epsilon \to 0$ the least-squares solution to the problem (5) converges to the optimal solution to the initial problem (2). The choice of the weighting coefficient ϵ did not pose a problem in the examples undertaken. The constraint $\epsilon t = 0$ enables one to stabilize the solution procedure. This is an important difference when compared with the big M method in linear programming, where the choice of the penalty coefficient M is complicated.

Remark 3.4. The big M method, which is based on the use of a penalty function, is deployed in linear programming. We will hereby name the method proposed the "little ϵ" " method. This method is especially suitable for solving problems with degenerate basis (see Examples 3.1, 3.2) and also when a feasible initial solution is not known. While solving practically viable problems, it has become clear that finding the initial solution is most computationally intensive task.

Example 3.1.

$$z = x_1 + x_2 + x_3 + x_4 \to max$$

$$(1+d)x_1 + x_2 + x_3 + x_4 \leq 4 + d$$
$$x_1 + x_3 + x_4 \quad \leq 3$$
$$x_1 + x_4 \quad \leq 2$$
$$x \geq 0.$$

The maximum value of the objective function is $z_{max} = 4 + d$, if $d > 0$. This is obtained in the case of one basis variable $x_2 = 4 + d$ as well as for basis consisting x_2, x_3 or x_2, x_4 or x_2, x_3, x_4. Additionally, if $x_3 + x_4 = 2$,then the objective function will achieve the maximum for $x_2 = 2 + d$.

We solved the problem for $d = 0,00001$ by transforming constraints to equalities by using slack variables . We varied the regularization parameter ϵ, $0 \leq \epsilon \leq 1$. The variables were activated in the order u_2, t, u_6, u_7. The last two correspond to slack variables. When we solved problem (5) for $\epsilon = 0$ (without the constraint $\epsilon t = 0$), $x_2* = 4,00001$ was obtained. When we set $\epsilon = 0,01$, we obtained $x_2* = 4,00003$. Even for a comparatively big $\epsilon = 1$ we obtained $x_2* = 4,2500$. The optimal values of variable change little when $0 \leq \epsilon \leq 0,001$.

Example 3.2.

$$z = 0,75x_1 - 150x_2 + 0,02x_3 - 6x_4 \to max$$
$$0,25x_1 - 60x_2 - 0,04x_3 + 9x_4 \leq 0$$
$$0,5x_1 - 90x_2 - 0,02x_3 + 3x_4 \leq 0$$
$$x_3 \quad \leq 1$$
$$x \geq 0.$$

While solving this problem with the simplex method, an infinitive cycle may accur, see/7/, $x* = (0,04;0;1;0)^T, z_{max} = 0,05$. Variables were activated in sequence u_1, u_3, t, u_5, while solving problem (5). If parameter $\epsilon = 0$, then optimal solution had accuracy of 10^{-14}. For $\epsilon = 0,01$ $x* = (0,046;0;1,0001;0)^T$.

Remark 3.5. Let us take a look at a problem, where all right hand sides are zero. We formulate problem (5).

$$(c, u) = 1, \quad Au = 0, \tag{7}$$
$$u \geq 0.$$

If this problem has a solution, which satisfies all equations, then by multiplying this solution with any positive number, we see that the objective function is unbounded. In the other case, the system (7) does not have an exact solution, $z_{max} = 0, x* = 0$.

Example 3.3.

$$z = 2x_1 + 3x_2 - x_3 - 12x_4 \to max$$
$$-2x_1 - 9x_2 + x_3 + 9x_4 + x_5 \ = 0$$
$$1/3x_1 + x_2 - 1/3x_3 - 2x_4 \ \ + x_6 = 0$$
$$x \geq 0.$$

This is a total cycle problem of A.Tucker, see /6/. The corresponding problem (5) has a solution, which exactly satisfies all constraints for each $\epsilon, 0 \leq \epsilon \leq 1$ $u* = (1, 0, 1, 0, 1, 0, 0)^T, t* = 0$. Thus the objective function of initial problem is unbounded for $c_1 > 1$. For example, if $c_1 = 0, 5$, then constraints of dual problem are satisfied, $z_{max} = w_{min} = 0, x* = 0$. The dual problem has a single feasible solution $y = (0, 3)$ if $c_1 \leq 1$, in the other case the dual problem is contradictory.

Example 3.4. In paper /9/ a curve fitting problem is solved. A set on corresponding values (x_i, y_i) are collected, $i = 1, ..., 19$. Fit the "best" straight line $y = bx+a$ to this set of data points. The objective is to minimize the sum of absolute deviations of each observed value of y from the value predicted by the linear relationship.

$$z = \sum u_i + \sum v_i \to min$$
$$bx_i + a + u_i - v_i = y_i$$
$$u \geq 0, v \geq 0, i = 1, ..., 19.$$

The solution to this problem is line $y = 0, 6375x + 0, 5812$. We changed the sign of the objective function and added all constraints to it. We solved the problem for a number of ϵ, $0, 001 \leq \epsilon \leq 1.$. The most inexact solution was obtained for $\epsilon = 1$, $y = 0, 6386x + 0, 5864$. But the number of steps undertaken decreased for growing ϵ values.

Finally we put forth the formal description of the small ϵ method.

1. **If** rhs $b = 0$, **then**
... solve NNLS problem (5)
..... **If** $r^2 = min[(1 - (c, u))^2 + \| Au \|^2, u \geq 0] = 0$ **then**
..... stop, the objective function z is unbounded
..... **else** $z_{max} = 0, x* = 0$.
end If

2. Let us transform the objective function so that $z_{max} > 1$ (see Remark 3.1)
3. Solve NNLS problem (5)
4. **If** $t* > 0$ and $x* = u * /t*$ satisfies $Ax* = b$ **then**
...$x*$ is an approximate solution to the problem (2), stop
...**else** the problem (2) has any feasible solution, stop
5. **If** $t* = 0$ and $R^2 = min[\| Au - b \|^2, u \geq 0] > 0$ **then**
... the problem (2) has any feasible solution, stop
6. The objective function z is unbounded.
7. The problem (2) is solved.

Example 3.5.

$$z = 2x_1 - x_2 \rightarrow max$$

$$x_1 - x_2 + x_3 \quad = 1$$

$$x_1 - x_2 \quad - x_4 = 2$$

$$x \geq 0.$$

The initial as well as the dual problems are contradictory. The least-squares solution to the problem (5) is $u* = (1, 1, 0, 0)^T, t* = 0$, which infers that the dual problem is contradictory. In such a case the theorem of duality implies that the initial problem is contradictory or the objective function is unbounded. If we solve only the right hand side of the second constraint by taking $b_2 = 0$, then the constraints are no longer contradictory and the objective function is unbounded, see steps 5, 6.

4 Conclusion and future work

In this paper an algorithm based on non-negative least-squares method is proposed. That algorithm is primarily suitable for degenerate problems, as well as for linear programming problems for which the initial solution is not known. The NNLS problem my be solved also using Matlab's built-in solver. In the future the $QR-$ decomposition will have to be used for solving sparse linear programming problems, see Björck /2/.

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