ON A RESULT CONCERNING REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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ABSTRACT. Let EN(T; Φ' , Φ'') denote the average number of real roots of the random trigonometric polynomial

$$T=T_n(\theta,\omega)=\sum_{K=1}^{n} a_K(\omega)\cos k\theta$$

In the interval (Φ' , Φ''). Clearly , T can have at most 2n zeros in the interval (0, 2π) .Assuming that $a_k(\omega)$s to be mutually independent identically distributed normal random variables , Dunnage has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions $(T_n(\theta\omega))$ have $\frac{2n}{\sqrt{3}}+O\left(n^{11/13}(\log n)^{3/13}\right)$ zeros when n is large. We consider the same family of trigonometric polynomials and use the Kac_rice formula for the expectation of the number of real roots and obtain that

$$EN(T;\ 0,\ 2\pi) \sim \frac{2n}{\sqrt{6}}+O(\log n)$$

This result is better than that of Dunnage since our constant is $(1/\sqrt{2})$ Times his constant and our error term is smaller . the proof is based on the convergence of an integral of which an asymptotic estimation is obtained .

1991 Mathematics subject classification (amer. Math. Soc.): 60 B 99.

Keywords and phrases: Independent, identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots, domain of attraction of the normal law, slowly varying function.

1 . Introduction

Let N(T ; Φ' , Φ'') be the number of real zeros of trigonometric polynomial $T = T_n(\Phi, \omega) = \sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta$ **(1)**

In the interval (Φ' , Φ'') where the coefficients $a_k(\omega)$ are mutually independent random variables identically distributed according to the normal law ; $b_k=k^p$ are positive constants and when multiple zeros are counted only once . Let EN (T ; Φ' , Φ'') denote the expectation of N (T ; Φ' , Φ''). Obviously , T_n (Φ, ω) can have at most 2n most zeros in the interval (0 , 2π).**Dunnage [1]** has shown that *in the interval* $0 \le \theta \le 2\pi$ *all save a certain exceptional set of the functions* $T_n(\theta, \omega)$ *have*

$$\frac{2n}{\sqrt{3}} + O\left(n^{11/13} (\log n)^{3/13} \right)$$

zeros when n is large . The measure of the exceptional set does not exceed (logn)$^{-1}$. subsequently ,**Das[2]** and **Qualls [3]** have obtained similar results. In this note our purpose is to show that it is possible to obtain a still lower estimate for the expectation of the number of real roots of (1) by using the method of **Loggan & shepp [4].** We show that

$$EN (T ; 0 , 2\pi) \sim \frac{2n}{\sqrt{6}} + O(\log n)$$

This result is better than that of Dunnage since our constant is(1 /√2) times his constant and our error term is smaller.

2 . The Approximation for EN (T ; 0 , 2π)

Let L (n) be a positive-valued function of n such that L(n) and n/ L(n) both approach infinity with n . We take $\in$=L(n)/n throughout.

Outside a small exceptional set of ω, T_n(θ ,ω) has a negligible number of zeros in each of the intervals (0, $\in$) ,(π-$\in$, π+$\in$) and (2π-$\in$,2π). By periodicity , of zeros in each of intervals (0, $\in$) and (2π-$\in$,2π) is the same as number in (-$\in$,$\in$). We shall use the following lemma , which is due to Das [2] . *Lemma.* The probability that T_n(θ ,ω) has more than 1 + (2 / log 2)(logn+2n$\in$) Zeros in ω-$\in$≤ θ ≤ ω+$\in$ does not exceed 2 exp(-n$\in$) .This lemma is due to **Das[2]** , in the special case $D_n=\sum b_n$ = n.The expected number of zeros of T in the interval (Φ' , Φ'') is given by the Kac_Rice formula

$$EN(T;\ \Phi', \Phi'') = \int_{-\phi'}^{\phi''} d\theta \int_{-\infty}^{\infty} |\eta| p(0,\eta) d\eta \qquad (2)$$

Where the probability density $p(\xi,\eta)$ T=ξ and T'=η is given by the Fourier inversion formula

$$p(\xi,\eta) = \frac{1}{(2\Pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-i\xi y - i\eta z)\phi(y,z)dydz$$

$\phi(y,z) = E\{\exp(iTy + iT'z)\}$ being the characteristic function of the combined variable (T , T'). In our case ,we have

$$T = \sum_{K=1}^{n} a_K(\omega)\cos k\theta \qquad T' = -\sum_{K=1}^{n} k a_K(\omega)\sin k\theta$$

$$\phi(y,z) = \exp\left\{-\sum_{K=1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}$$

$$p(0,\eta) = \frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \exp(1 - i\eta z)\exp\left\{-\sum_{K=1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}dy$$

for ε > 0 ,

$$\int_{-\infty}^{\infty} |\eta|\exp(-\in|\eta|)p(0,\eta)d\eta = \mathrm{Re}\frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)d\eta\int_{-\infty}^{\infty}dz$$

$$\int_{-\infty}^{\infty}\exp(-i\eta z)\exp\left\{-\sum_{1}^{n}(y\cos\theta - zk\sin k\theta)^2\right\}dy$$

$$= \mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in -iz)^2} + \frac{1}{(\in +iz)^2}\right\}$$

$$\times\exp\left\{-\sum_{1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}dy \qquad (3)$$

where Re stands for the real part.

Here , if we allow coskθ , ksinkθ to be arbitrary , that is we take each of them to be constant in k ,then the probability density p(ξ, η) Of ξ=T(θ) = AX and η= T'(θ) =BX , say , degenerates and we get from (3) the following identity , valid for non-zero A and B which can be chosen suitably .

$$0 = \mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in -iz)^2}+\frac{1}{(\in +iz)^2}\right\}\exp\left\{-(Ay-Bz)^2\right\}dy \qquad \textbf{(4)}$$

Subtracting (4) from (3) we get

$$\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)p(0,\eta)d\eta$$

$$= \mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in -iz)^2}+\frac{1}{(\in +iz)^2}\right\}$$

$$\times\left\{\exp\left\{-\sum_{1}^{n}(y\cos k\theta - zk\sin k\theta)^2\right\}-\exp\left(-(Ay-Bz)^2\right)\right\}dy$$

$$= \mathrm{Re}\frac{1}{\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}z\left\{\frac{1}{(\in -iz)^2}+\frac{1}{(\in +iz)^2}\right\}$$

$$\times\left\{\exp\left(-Gz^2\right)-\exp(-Hz^2)\right\}du \qquad \textbf{(5)}$$

by transforming the integrals putting y = - uz or y = uz and denoting

$$G=\sum_{K=1}^{n}\left(u\cos k\theta + k\sin k\theta\right)^2$$

And H=(Au + B)^2

Now using the identity **(Logan and shepp[4]** , for α =2),

$$\int_{0}^{\infty}\left\{\exp\left(-Hz^2\right)-\exp\left(-Gz^2\right)\right\}\frac{dz}{z}=\frac{1}{2}\log(G/H)$$

In the limit as ε⟶0 we obtain from (5) that

$$\int_{-\infty}^{\infty}|\eta|p(0,\eta)d\eta=\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta + k\sin k\theta)^2}{(Au+B)^2}\right\}du \qquad \textbf{(6)}$$

The double integral appearing in (5) is dominated by a decreasing exponential function. So the involved integrals are uniformly convergent on any interval. Since the integral on the right side of (6) converges , we conclude that both the passage to the limit by letting ε⟶0 and the subsequent change of the order of integration to produce the equation (6) are justified.

3 . Estimation of the integral of equation (6)

In this section we obtain an asymptotic estimation for the integral

$$I = \int_{-\infty}^{\infty} \log\left\{ \frac{\sum_{K=1}^{n} (u\cos k\theta + k\sin k\theta)^2}{(Au+B)^2} \right\} du$$

Where A and B are fixed non-zero real numbers . this integral exists in general as a principal value i.e.

$$\lim_{R\to\infty} \int_{-R}^{R} \ldots \quad \text{, if} \qquad A^2 = \sum_{K=1}^{n} \cos^2 k\theta$$

Let $B^2 = \sum_{K=1}^{n} k^2 \sin^2 k\theta$ And $C^2 = \sum_{K=1}^{n} k \cos k\theta \sin k\theta$

As in **Das[2 ,pp.727]** we have for

$$A^2 = \frac{1}{2}\{1 + O(1/\log n)\}n = \frac{1}{2}Sn$$

$$B^2 = \frac{1}{6}\{1 + O(1/\log n)\}n^3 = \frac{1}{6}Sn^3$$

and $$C^2 = O(n^2/\log n) = \frac{\beta n^2}{\log n} \quad \text{, (} \beta \text{ = constant) ,}$$

Taking L(n) = logn .

We have always by Cauchy's inequality , $AB \geq C^2$. In what follows we will assume that $AB > C^2$. This happens if θ does not take values from the set {0,± π, ±2π,… } . In fact ,

$$A^2B^2 - 2C^4 = \frac{S^2n^4}{12}\left\{1 - \frac{24\beta^2}{S^2(\log n)}\right\} \cong \frac{S^2n^4}{12} = A^2B^2 \qquad \textbf{(7)}$$

So that

$$I = \int_{-\infty}^{\infty} \log\left\{ \frac{\sum_{K=1}^{n} (u\cos k\theta + k\sin k\theta)^2}{(Au+B)^2} \right\} du$$

$$= \int_0^{\infty} \log\left\{ \frac{\left(A^2u^2 + B^2\right)^2 - 4u^2C^4}{A^4u^4 + B^4 - 2u^2A^2B^2} \right\} du$$

$$\cong \int_0^{\infty} \log\left\{ \frac{A^4u^4 + B^4 + 2u^2A^2B^2}{A^4u^4 + B^4 - 2u^2A^2B^2} \right\} du \text{ by (7)}$$

= I ' , say **(8)**

$$= \int_0^{\infty} \log\left\{ \frac{1+x}{1-x} \right\} du \text{ , writing } x = (2u^2A^2B^2) / (A^4u^4 + B^4)$$

$$= \int_0^{\infty} \log\left\{ 1 - \frac{4x}{(1+x^2)} \right\}^{-1/2} du$$

$$= \frac{1}{2} \int_0^{\infty} \{ -\log(1-z) \} du \text{ , putting } z = 4x / (1+x^2) .$$

now $x \to 0^+$ as $u \to 0$ or ∞. But $x > \varepsilon > 0$, if $\varepsilon A^4 u^4 - 2u^2A^2 B^2 + \varepsilon B^4 < 0$, which occurs for all u in the interval (d1 {O(n^2) / $\sqrt{\varepsilon}$ } –d2) ,where d1 , d2 are functions of ε tending to zero as $\varepsilon \to 0$. Thus for all u in the interval (0 , ∞) we can safely assume that $\varepsilon = 1 / n$,and x= {1 / L(n)} , where n is tending to infinity .

Thus

$$I' > 2\int_0^{\infty} \frac{x}{(1+x)^2} du$$

$$= 2\int_0^{\infty} \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 x du$$

$$= 4\left\{ 1 - \frac{1}{L(n)+1} \right\}^2 \int_0^{\infty} \frac{u^2A^2B^2}{A^4u^4 + B^4} du$$

$$= \frac{4B}{A} \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 \int_0^{\infty} \frac{v^2}{v^4+1} dv$$

$$= \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 . \frac{2\Pi n}{\sqrt{6}} \qquad \textbf{(9)}$$

Again

$$I' < \frac{1}{2}\int_0^{\infty} \frac{z}{1-z}du = \frac{1}{2}\int_0^{\infty} \frac{4x}{(1-x)^2}du$$

$$= 2\int_0^{\infty} \left\{1 - \frac{1}{L(n)-1}\right\}^2 x\,du$$

$$= \left\{1 - \frac{1}{L(n)+1}\right\}^2 . \frac{2\Pi n}{\sqrt{6}} \qquad (10)$$

now from (9) and (10) $I' \sim \frac{2\Pi n}{\sqrt{6}}$ (11)

And from (8) and (11) $I \sim \frac{2\Pi n}{\sqrt{6}}$ (12)

4 . EN (T ; Φ' , Φ'')

From (2) , (6) and (12) , we obtain EN (T ; Φ' , Φ'') = $\frac{(\Phi''-\Phi')n}{\sqrt{6}}$

In view of our choice of A , B and C

EN (T ; π+ε , 2π-ε) = EN (T ; ε , π-ε)

Again , by the lemma , we have

EN (T ; 0 , ε) + EN (T ; π- ε , π+ε) + EN (T ; 2π-ε , 2π)

= EN (T ; π+ε , 2π-ε) ≤ 2 { 1 + (2 / log 2)(log n + 2nε) }

Now choosing ε = (log n) / n , the desired result follows .

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