# Dynamical Analysis of a Discrete Fractional Order Prey-Predator 3 – D System

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## Dedicated to Dr.Major.M.ReniSagayaraj

**Abstract**. In this paper, we propose and discuss the dynamical nature of discrete fractional order prey – predator systems. A discretization process is applied to obtain itsdiscrete version. Fixed points and the asymptotic stability are investigated.Chaotic attractor, bifurcation and chaos for different values of the fractional order parameter arediscussed.

2000 Mathematics Subject Classification.26A33, 39A11, 39A12.

Key words and phrases.Fractional Order, discretization, Fixed points, Prey – Predator, stability, Bifurcation.

## 1. INTRODUCTION

Fractional derivatives have a long mathematical history and it is a 300 year old topic. Interdisciplinary applications can be elegantly modeled with the help of the fractionalderivatives. During the last decade fractional calculus has attracted much more attention of physicists and mathematicians.Fractional order differential equations are naturally related to systems with memorywhich exists in most biological systems. Mathematical modeling in population biology provides new aspects in understanding the interaction between species. A large number of research work has been done on modeling the dynamics of interaction among species, it has been restricted to integer order differential/difference equations. In recent years, it has turned out that many phenomena in different fields can be described very successfully by the models using fractional order differential equations. We begin by giving the definitions and some properties of fractional order integrals and derivatives [7].

**Definition 1**. The fractional integral of order $\beta \in R^+$ of the function $f(t), t > 0$, is defined by

$$I^{\beta} f(t) = \int_0^1 \frac{(t-s)^{(\beta-1)}}{\Gamma(\beta)} f(s) ds.$$

and the fractional derivative order $\alpha \in (n-1, n)$ of $f(t), t > 0$, is defined by

$$D^{\alpha} f(t) = I^{n-a} D^{\alpha} f(t),\ D = \frac{d}{dt}$$

In addition, the following results are the main ones in fractional calculus. Let $\alpha, \beta \in R^{+}$.

- $I_{\alpha}^{\beta} : L^1 \to L^1$, andif $f(x) \in L^1$, then$I_{\alpha}^{\gamma} I_{\alpha}^{\beta} = I_{\alpha}^{\gamma+\beta} f(x)$.
- $\lim_{\beta \to n} I_{\alpha}^{\beta} f(x) = I_{\alpha}^{n} f(x)$ uniformly on $[a,b], n = 1,2,3,\cdots$ where $I_{\alpha}^{1} f(x) \int_0^x f(s) ds$.
- $\lim_{\beta \to 0} f(x) = f(x)$ weakly.
- If $f(x)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 0} D_a^{\alpha} f(x) = \frac{df(x)}{dx}$.

## 2. DISCRETIZATION PROCESS

In [3,9], a discretization process is introduced to discretize the fractional order logistic differential equations. When the fractional order parameter $\alpha \to 1$, Euler's discretizationmethod is obtained. Here, we are interested in applying the discretization method to a system of differential equations describing the prey predator interaction. Let $\alpha \in (0,1)$and consider the differential equation of fractional order

$$\begin{aligned} &D^{\alpha} x(t) = f\left(x(t)\right), t > 0 \\ &x(0) = x_0, t \le 0 \end{aligned} \qquad (1)$$

The corresponding equation with a piecewise constant argument

$$\begin{aligned} &D^{\alpha} x(t) = f\left(x\left(r\left[\frac{t}{r}\right]\right)\right), t > 0 \\ &x(0) = x_0, t \le 0 \end{aligned} \qquad (2)$$

Let $t \in [0,r)$, then $\frac{t}{r} \in [0,1)$. So we get $D^{\alpha} x(t) = f(x_0),\ t \in [0,1)$.

Thus$x_1(t) = x_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f(x_0)$.

Let $t \in [r, 2r)$, then$\frac{t}{r} \in [1,2)$. So we get$D^{\alpha} x(t) = f(x_1)), t \in [r, 2r)$.

Thus$x_2(t) = x_1(r) + \frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)} f(x_1(r))$.

Let $t \in [2r, 3r)$, then $\frac{t}{r} \in [2,3)$. So we get $D^{\alpha} x(t) = f(x_2(2r)),\ t \in [2r, 3r)$

Thus $x_3(t) = x_2(2r) + \frac{(t-2r)^{\alpha}}{\Gamma(1+\alpha)} f(x_2(2r))$.

Repeating the process, we get when $t \in [nr, (n+1)r)$, then $\frac{t}{r} \in [n, n+1)$. So we get

$$D^{\alpha} x(t) = f(x_n(nr)),\ t \in [nr, (n+1)r)$$

Thus

$$x_{(n+1)}(t) = x_n(nr) + \frac{(t-nr)^{\alpha}}{\Gamma(1+\alpha)} f(x_n(nr))$$

## 3. MODEL DESCRIPTION AND DISCRETIZATION

In 1926, Volterra came up with a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. They were proposed independently by Alfred J.Lotka in 1925 [1, 2,4,5,6]. The well-known system is as follows:

$$x' = ax - bxy : y' = -cy + dxy$$

With a = b = c = d = 1, [8] discusses he dynamical properties of the fractional order LotkaVolterra equation

$$D_0^{\alpha} + x = x(1-y);$$

$$D_0^{\alpha} + y = y(-1+x).$$

Here we are concerned with the fractional order Prey Predator system given by

$$\begin{aligned} D^{\alpha} x(t) &= ax(t) - bx(t)z(t) \\ D^{\alpha} y(t) &= cy(t)\left[1 - y(t)\right] - dy(t)z(t) \\ D^{\alpha} z(t) &= ex(t)z(t) + fy(t)z(t) + gz(t) \end{aligned} \tag{3}$$

where $\alpha$ is the fractional order. Now we are actually interested in discretizing fractional order prey predator system given in the form

$$D^{\alpha}x(t)=ax\left(r\left[\frac{t}{r}\right]\right)-bx\left(r\left[\frac{t}{r}\right]\right)z\left(r\left[\frac{t}{r}\right]\right)$$
$$D^{\alpha}y(t)=cy\left(r\left[\frac{t}{r}\right]\right)\left[1-y\left(r\left[\frac{t}{r}\right]\right)\right]-dy\left(r\left[\frac{t}{r}\right]\right)z\left(r\left[\frac{t}{r}\right]\right) \tag{4}$$
$$D^{\alpha}z(t)=ex\left(r\left[\frac{t}{r}\right]\right)z\left(r\left[\frac{t}{r}\right]\right)+fy\left(r\left[\frac{t}{r}\right]\right)z\left(r\left[\frac{t}{r}\right]\right)-gz\left(r\left[\frac{t}{r}\right]\right)$$

with initial condition $x(0) = x_0, y(0) = y_0, z(0) = z_{0.}$ The proposed discretization method has the following steps.

(1.) Let $t \in [0, r)$, then $\frac{t}{r} \in [0,1)$. So we get

$$D^{\alpha}x(t) = ax_0 - bx_0z_0, D^{\alpha}y(t) = cy_0[1 - y_0] - dy_0z_0, D^{\alpha}z(t) = ex_0z_0 + fy_0z_0 - gz_0$$

and the solution of (4) by

$$x_1(t) = x_0 + I^{\alpha}ax_0 - bx_0z_0 = x_0 + (ax_0 - bx_0z_0)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$
$$y_1(t) = y_0 + I^{\alpha}cy_0[1-y_0] - dy_0z_0 = y_0 + \left(cy_0\left[1-y_0\right] - dy_0z_0\right)\frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}$$
$$z_1(t) = z_0 + I^{\alpha}ey_0z_0 + fy_0z_0 - gz_0 = z_0 + (ey_0z_0 + fy_0z_0 - gz_0)\frac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}$$

(2.)Let $t \in [r, 2r)$, then $\frac{t}{r} \in [1,2)$. So we get

$$D^{\alpha}x(t) = ax_1 - bx_1z_1, D^{\alpha}y(t) = cy_1[1 - y_1] - dy_1z_1,\ D^{\alpha}z(t) = ex_1z_1 + fy_1z_1 - gz_1$$

and the solution of (4) by

$$x_2(t) = x_1(r) + I^{\alpha}ax_1(r) - bx_1(r)z_1(r)$$
$$= x_1(r) + (ax_1(r) - bx_1(r)z_1(r))\frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)}$$
$$y_2(t) = y_1(r) + I^{\alpha}cy_1(r)[1-y_1(r)] - dy_1(r)z_1(r)$$

$$= y_1(r) + \left(cy_1(r)\left[1 - y_1(r)\right] - dy_1(r)z_1(r)\right)\frac{(t-r)^\alpha}{\Gamma(1+\alpha)}$$

$$z_2(t) = z_1(r) + I^\alpha ey_1(r)z_1(r) + fy_1(r)z_1(r) - gz_1(r)$$

$$= z_1(r) + (ey_1(r)z_1(r) + fy_1(r)z_1(r) - gz_1(r))\frac{(t-r)^\alpha}{\Gamma(1+\alpha)}$$

Repeating the process, we can easily deduce that the solution of (4) is given by

$$x_{n+1}(t) = x_n(nr) + \frac{(t-nr)^\alpha}{\Gamma(1+\alpha)}[ax_n(nr) - bx_n(nr)z_n(nr)]$$

$$y_{n+1}(t) = y_n(nr) + \frac{(t-nr)^\alpha}{\Gamma(1+\alpha)}[\left(cy_n(nr)\left[1 - y_n(nr)\right] - dy_n(nr)z_n(nr)\right)]$$

$$z_{n+1}(t) = z_n(nr) + \frac{(t-nr)^\alpha}{\Gamma(1+\alpha)}[ey_n(nr)z_n(nr) + fy_n(nr)z_n(nr) - gz_n(nr)]$$

Hence the discrete version is

$$\begin{aligned} x_{n+1} &= x_n + \frac{r^\alpha}{\Gamma(1+\alpha)}[ax_n - bx_nz_n] \\ y_{n+1} &= y_n + \frac{r^\alpha}{\Gamma(1+\alpha)}[cy_n[1-y_n] - dy_nz_n] \\ z_{n+1} &= z_n + \frac{r^\alpha}{\Gamma(1+\alpha)}[ex_nz_n + fy_nz_n - gz_n] \end{aligned} \tag{5}$$

## 4. FIXED POINT AND STABILITY WITH NUMERICAL SIMULATIONS

Now we analyze the stability of the fixed points of the system (5) which has the following fixed points:

$F_0 = (0,0,0)$, Trivial point. $F_1 = (0,1,0)$, Axial point.

$F_2 = \left(\frac{g}{e}, 0, \frac{a}{b}\right)$, Axial point. $F_3 = \left(0, \frac{g}{f}, \frac{c(f-g)}{df}\right)$, Axial point.

$F_4 = \left(\frac{adf}{bce} - \frac{(f-g)}{e}, \frac{(bc-ad)}{bc}, \frac{a}{b}\right)$, Interior point.

By considering the Jacobian matrix for interior fixed point and calculating their Eigen values, we can investigate the stability of the interior fixed point based on the roots of the system characteristic equation. Linearizing system (5) about $F_4$ yield the characteristicequation:

$$P(\lambda) = \lambda^3 + \lambda^2\left[s\left(c - \frac{ad}{b}\right) - 3\right] + \lambda\left[3 + \left[2s\left(\frac{ad}{b} - c\right)\right] + s^2\left[\left(\frac{ad}{b} - c\right)\left(\frac{af}{c} - \frac{adf}{bc}\right) + ag\right]\right]$$
$$+ s\left(c - \frac{ad}{b}\right) - 1 + s^2\left[a(f-g) - \frac{adf}{b}\left[\frac{a}{c} - \frac{ad}{bc} + 1\right]\right]$$
$$+ s^2\left[\left(c - \frac{ad}{b}\right)\left[\frac{a^2df}{bc} - a(f-g)\right]\right] = 0$$

where $s = \frac{r^\alpha}{\Gamma(1+\alpha)}$. Let

$$a_1 = \left[s\left(c - \frac{ad}{b}\right) - 3\right]$$

$$a_2 = \left[3 + \left[2s\left(\frac{ad}{b} - c\right)\right] + s^2\left[\left(\frac{ad}{b} - c\right)\left(\frac{af}{c} - \frac{adf}{bc}\right) + ag\right]\right]$$

$$a_3 = s\left(c - \frac{ad}{b}\right) - 1 + s^2\left[a(f-g) - \frac{adf}{b}\left[\frac{a}{c} - \frac{ad}{bc} + 1\right]\right]$$
$$+ s^2\left[\left(c - \frac{ad}{b}\right)\left[\frac{a^2df}{bc} - a(f-g)\right]\right]$$

From the Jury test, if $P(1) > 0, P(-1) < 0$, and $a_3 < 1, |b_3| > b_1, c_3 > |c_2|$, where $b_3 = 1 - a_3^2, b_2 = a_1 - a_3a_2, b_1 = a_2 - a_3a_1, c_3 = b_3^2 - b_1^2$, and $c_2 = b_3b_2 - b_1b_2$, then the rootsof P( $\lambda$) satisfy $\lambda$< 1 and thus $F_4$ is asymptotically stable. Suppose P(1) < 0; then $F_4$ isunstable.

Numerical simulations are helpful in analyzing the phase diagrams of dynamical system depending up on parameters. Numerical study of Fractional order discrete dynamicalsystems gives an insight in to dynamical characteristics. In this section, we present thetime plots for *x(t)*, *y(t)*, *z(t)*, phase portraits and bifurcation diagrams for the system (5).Dynamic behavior of the system (5) about the interior fixed points under different sets ofparameter values are presented.

**Example 1.** Let us consider the parameters with values *r = 0.1*,*a = 0.11*, *b = 0.21*, *c =0.8*, *d = 0.1*, *e = 0.4*, *f = 0.12*, *g = 0.28*, and the initial conditions are *x = 0.5*, *y = 0.4*, *z = 0.2*and the fractional derivative order *α = 0.95;* Applying Jury test we get *P(1) = 0.0003 >0*, *P(-1) = -7.6572 < 0* and $a_3$ = *-0.9144 < 1*, thus $F_4$is asymptotically stable see fig – 1.

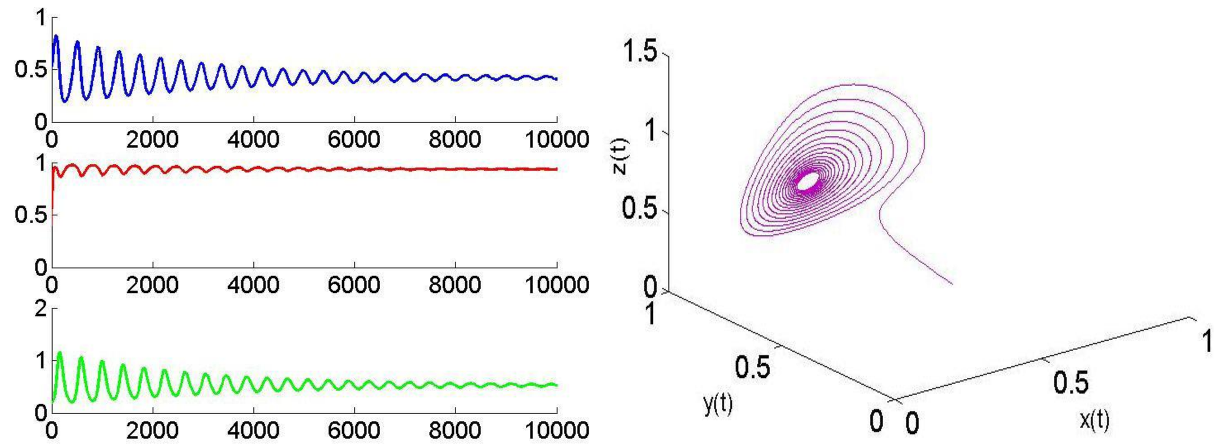


Figure 1. Time series and Phase diagram of Fixed point $F_4$ with Stability of system (5)

**Example 2.** Let us consider the parameters with values *r = 0.1, a = 0.11, b = 0.21, c =0.8, d = 0.1, e = 0.4, f = 0.12, g = 0.28,* and the initial conditions are *x = 0.5, y = 0.4, z = 0.2* and the fractional derivative order *α = 0.95,* Applying Jury test we get *P(1) = - 0.0012 < 0*,thus $F_4$is Unstable see fig – 2.

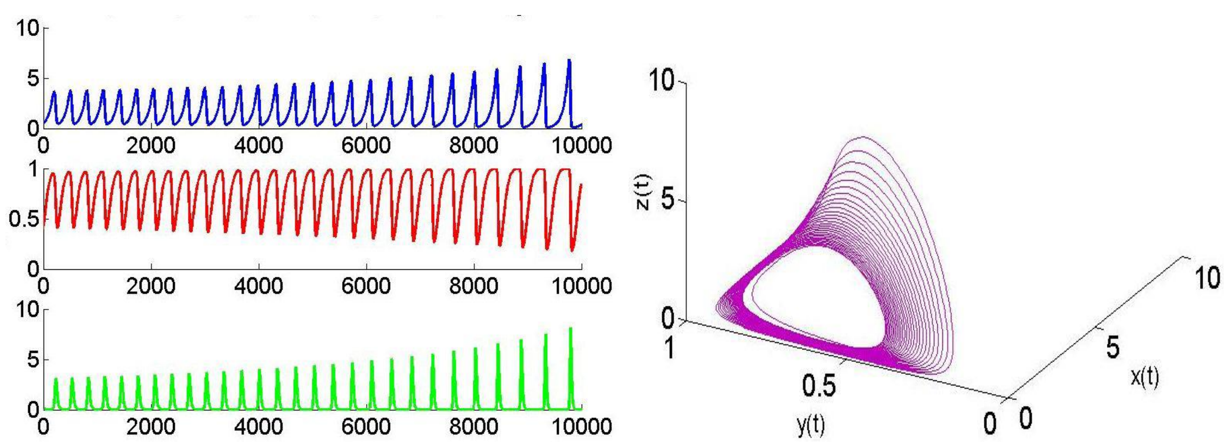


Figure 2. Time series and Phase diagram of Fixed point $F_4$with Unstability of system (5)

Bifurcation diagrams provide information about abrupt changes in the dynamics, see figure – 3. The parametric values at which these changes occur are called bifurcation points. They provide information about the dependence of the dynamics on a certain parameter. If the qualitative change occurs in a neighborhood of an equilibrium point or periodic solution, it is called a local bifurcation. Any other qualitative change that occurs is considered as a global bifurcation. Here consider the parameters with values *r = 0 – 0.3, a = 0.11, b =0.21, c = 0.35, d = 0.28, e = 0.14, f = 0.12, g = 0.2*, the initial conditions are *x = 0.5, y =0.4, z = 0.2* and the fractional derivative order *α = 0.99.*

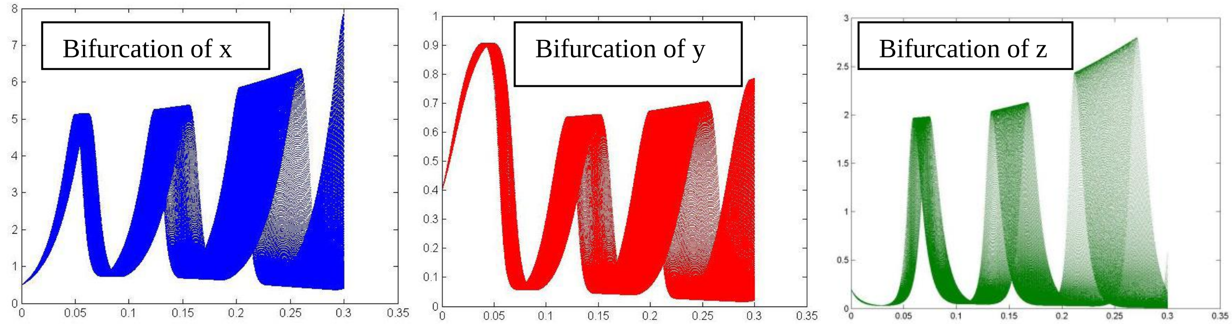


Figure 3. Bifurcation Diagram for Prey – Predator system of (5)

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