Conquering Basic Properties of Even Perfect Numbers Using the Gun of Finite Series

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Abstract. Mathematicians have been fascinated for centuries by the properties and patterns of numbers. They have noticed that some numbers are equal to the sum of all of their factors (not including the number itself). Such numbers are called perfect numbers. Thus a positive integer is called a perfect number if it is equal to the sum of its proper positive divisors. The search for perfect numbers began in ancient times. The four perfect numbers 6, 28, 496, and 8128 seem to have been known from ancient times. In this paper, we will investigate some important basic properties of perfect numbers.

Key Words: Prime Numbers, Perfect numbers, and Triangular numbers.

1. Introduction and Background

Throughout history, there have been studies on perfect numbers. It is not known when perfect numbers were first studied and indeed the first studies may go back to the earliest times when numbers first aroused curiosity [6]. It is rather likely, although not completely certain, that the Egyptians would have come across such numbers naturally given the way their methods of calculation worked, where detailed justification for this idea is given [6]. Perfect numbers were studied by Pythagoras and his followers, more for their mystical properties than for their number theoretic properties [6]. Although, the four perfect numbers 6, 28, 496 and 8128 seem to have been known from ancient times and there is no record of these discoveries [6]. The First recorded mathematical result concerning perfect numbers which is known occurs in Eculid's Elements written around 300BC [6]

Theorem 1 . If 2^k-1 (k>1) is prime, then $n=2^{k-1}(2^k-1)$ is a perfect number.

Proof: We will show that n = sum of its proper factors.
We will find all the proper factors of $2^{k-1}\left(2^k-1\right)$, and add them.
Since 2^k-1 is prime, let $p=2^k-1$. Then $n = p(2^k-1)$

Let us list all factors of 2^{k-1} and other proper factors of n as follows .

Factors of $2k$-1	Other Proper Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-2}\,p$

Adding the first column, we get:

$$1+2+2^2+2^3 \ldots + 2^{k-3}+2^{k-2}+2^{k-1}$$
$$=2^k-1$$
$$=p$$

Adding the second column, we get:

$$p+2p+2^2\,p+2^3\,p\ldots+2^{k-4}\,p+2^{k-3}\,p+2^{k-2}\,p$$
$$=p\left(1+2+2^2+\ldots+2^{k-2}\right)$$
$$=\left(2^{k-1}-1\right)p$$

Now adding the two columns together, we get:

$$p+p\left(2^{k-1}-1\right)$$
$$=p\left(1+2^{k-1}-1\right)$$
$$=p\left(2^{k-1}\right)$$
$$=n$$

Hence. n is a perfect number.

Remark I: A question can be raised if k is prime by itself

$\Rightarrow$ $2^{k-1}(2^k-1)$ is a perfect number. The answer is negative as it will be easily shown that it does not work for k=11.

Corollary 1: : If 2^k-1 is prime, then $n=2^{k-1}+2^k+2^{k+1}\ldots+2^{2k-2}$ is a perfect number.

Proof: We have:

$$n = 2^{k-1} + 2^{k} + 2^{k+1} \ldots + 2^{2k-2} = 2^{k-1}\left(1 + 2 + 2^{2} + 2^{3} \ldots + 2^{k-1}\right)$$
$$n = 2^{k-1}\left(2^{k} - 1\right)$$
$\Rightarrow$ n is a perfect number by Theorem 1.

Remark III: Every even perfect number n is of the form
$n = 2^{k-1}(2^{k}-1)$. We will not prove this, but we will accept and use it.
So, the converse to Theorem 1 is also true. This is called Euler's Theorem.

Next we will show how **Remark III** applies to the first four perfect numbers. Note that:

$$6 = 2 \cdot 3 = 2^{1}\left(2^{2} - 1\right) = 2^{2-1}\left(2^{2} - 1\right)$$
$$28 = 4 \cdot 7 = 2^{2}\left(2^{3} - 1\right) = 2^{3-1}\left(2^{3} - 1\right)$$
$$496 = 16 \cdot 31 = 2^{4}\left(2^{5} - 1\right) = 2^{5-1}\left(2^{5} - 1\right)$$
$$8128 = 64 \cdot 127 = 2^{6}\left(2^{7} - 1\right) = 2^{7-1}\left(2^{7} - 1\right)$$

Theorem II. Every even perfect number n *is* a triangular number.

Proof: n is a perfect number $\Rightarrow$ n= $2^{k-1}(2^{k}-1)$ by Remark III. Hence,
n= $\dfrac{2^{k}(2^{k}-1)}{2} = \dfrac{(m+1)m}{2}$, where m=$2^{k}$-1 . Thus n is a triangular number.

Corollary II If T is a perfect number, then 8T +1 is a perfect square**.**

Proof: T is a perfect number $\Rightarrow$ T is a triangular number.
$\Rightarrow$ T=$\dfrac{(m+1)m}{2}$ for some positive integer m.
$\Rightarrow$ 8T+1 = $4\,m(m+1)+1$
= $4m^{2}+4m+1$
=$(2m+1)^{2}$

Next we will prove two important theorems .

Theorem III: The sum of the reciprocals of the factors of a perfect number is n is equal to 2.

Proof: Let $n = 2^{k-1}\,(2^{k}-1)$ where $p = 2^{k}-1$ and is prime. Let us list all the possible factors of n.

Factors of 2^{k-1}	Other Factors
1	p
2	$2p$
2^{2}	$2^{2}p$
2^{3}	$2^{3}p$
:	:
:	:

$2^{k\text{-}1}$ $2^{k\text{-}1}p$

Sum of reciprocals of factors of $2^{k\text{-}1}$

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}}{2\left(2^{k-1}\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}\right)}\ldots+\frac{1}{\left(2^{k-1}\right)}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-2}}{2^{k-1}}+\frac{2^{k-3}}{2^{k-1}}\ldots+\frac{1}{2^{k-1}}$$
$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}}$$
$$=\frac{2^k-1}{2^{k-1}}=\frac{p}{2^{k-1}}$$

Sum of reciprocals of other factors

$$\frac{1}{p}+\frac{1}{2p}+\frac{1}{2^2p}+\frac{1}{2^3p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}}{2\left(2^{k-1}p\right)}+\frac{2^{k-1}}{2^2\left(2^{k-1}p\right)}\ldots+\frac{1}{\left(2^{k-1}p\right)}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-2}}{2^{k-1}p}+\frac{2^{k-3}}{2^{k-1}p}\ldots+\frac{1}{2^{k-1}p}$$
$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}\ldots+1}{2^{k-1}p}$$
$$=\frac{2^k-1}{2^{k-1}p}=\frac{p}{2^{k-1}p}=\frac{1}{2^{k-1}}$$

Now the sums of reciprocals of all factors are equal to:

$$= \frac{p}{2^{k-1}} + \frac{1}{2^{k-1}}$$
$$= \frac{p+1}{2^{k-1}}$$
$$= \frac{2^k - 1 + 1}{2^{k-1}}$$
$$= \frac{2^k}{2^{k-1}} = 2$$

Example 1. Factors of 8128 are: 1, 2, 4, 8, 16, 32, 64,127, 254, 508, 1016, 2032, 4064, and 8128. Note that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{127} + \frac{1}{254} + \frac{1}{508} + \frac{1}{1016} + \frac{1}{2032} + \frac{1}{4064} + \frac{1}{8128}$$
$$= \frac{8128 + 4064 + 2032 + 1026 + 508 + 254 + 127 + 64 + 32 + 16 + 8 + 4 + 2 + 1}{8128}$$
$$= \frac{16256}{8128} = 2$$

The following shows application of Theorem III.

Corollary III . No power of a prime can be a perfect number.

Theorem IV: If n is a perfect number such that $n = 2^{k-1}(2^k-1)$, then the product of the positive divisor's of n is equal to n^k.

Proof: We list factors of n as in Theorem 2

Factors of 2^{k-1}	Other Factors
1	P
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-1}p$

Product of column 1 =

$$1 * 2 * 2^2 * 2^3 \ldots * 2^{k-1} = 2^{1+2+3\ldots+(k-1)} = 2^{\frac{k(k-1)}{2}}$$

Product of column 2 =

$$p \cdot 2p \cdot 2^2 p \ldots \cdot 2^{k-1} p$$
$$= p^k (1 \cdot 2 \cdot 2^2 \ldots 2^{k-1})$$
$$= p^k (2^{\frac{k(k-1)}{2}}),$$

Therefore the products of both columns are

$$= 2^{\frac{k(k-1)}{2}} \cdot p^k \cdot 2^{\frac{k(k-1)}{2}}$$
$$= 2^{k(k-1)} \cdot p^k$$
$$= (2^{k-1} \cdot p)^k$$
$$= n^k.$$

Example 2: Apply Theorem IV to $n = 28$

$n = 28 = 2^2(2^3-1)$ (Here $k = 3$)

Factors of 28 are 1, 2, 4, 7, 14, and 28

The product of the factors of 28 =

$$1 \cdot 2 \cdot 4 \cdot 7 \cdot 14 \cdot 28$$
$$= 28 \cdot 28 \cdot 28$$
$$= 28^3$$

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