The Amazing Mathematical Beauty of the Mulatu Numbers With Interesting Open Questions

Mulatu Lemma

Department of Mathematics

Savannah State University
Savannah, GA 31404
USA

Abstract: The Mulatu Numbers were introduced by Mulatu Lemma, Professor of Mathematics at Savannah State University**.** The numbers are sequences of numbers of the form: **4,1,5,6,11,17,28,45...** The numbers have wonderful and amazing properties and patterns.

In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & if \;\; n = 0; \\ 1 & if \;\; n = 1; \\ M_{n-1} + M_{n-2} & if \;\; n > 1. \end{cases}$$

In [1] and [2] some properties and patterns of the numbers were considered. In this paper, we investigate additional properties and patterns of these fascinating numbers. Many beautiful mathematical identities involving the Mulatu numbers in relation with the Fibonacci numbers and the Lucas numbers will be more explored.

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1. Introduction and Background

As given in [1] and [2], the Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu (M_n), **Fibonacci** (F_n) and **Lucas** (L_n) **Numbers**
(Tables 1 & 2)

Table 1

n:	0	1	2	3	4	5	6	7	8	9	10	11
M_n:	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	12	13	14	15	16	17	18	19	20
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1 : Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers [1], Fibonacci and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $M_n = L_n + 2F_{n-1}$.

(4) $F_{2n} = F_n L_n$

(5) $5F^2{}_n - L_n{}^2 = 4(-1)^{n+1}$

(6) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(7) $L_{n+1} = L_n + L_{n-1}$

(8) $F_{n+k} = F_{n-1}F_k + F_n F_{k+1}$

The Main Results.

Theorem 1. **Some Divisibility Properties of M.**

(a) If M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3_{n+1} - M^3_{n-1}$ is divisible by 9.

Proof: We repeatedly use $M_{n+1} = M_n + M_{n-1}$ in our proof:

(a) $M^2_{n+1} - M^2_{n-1}$
$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n\,(M_n + M_{n-1} + M_{n-1}) = M^2_n + 2M_nM_{n-1}$.

Now it is easy to see that if M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) $M^3_{n+1} - M^3_{n-1} = (M_{n+1} - M_{n-1})(M^2_{n+1} + M_nM_{n-1} + M^2_{n-1})$

$$= M_n\,(M^2_{n+1} + M_{n+1}M_{n-1} + M^2_{n-1})$$

$$= M_n((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2_{n-1})$$

$$= M_n(M^2_n + 3M_nM_{n-1} + 3M^2_{n-1})$$

$$= M^3_n + 3M^2_nM_{n-1} + 3M_nM^3_{n-1}$$

Hence M_n is divisible by 3 $\Rightarrow$ M^3_{n+1} - M^3_{n-1} is divisible by 9.

Theorem 2. (Expressing M in terms of F)

Let M_n and F_n be any Mulatu and Fibonacci Numbers respectively. Then we have:

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}$$

Proof: We use induction on n.

(1) When n = 0, the formula is true as $M_0 = F_{-3} + F_{-1} + F_2$ and using $F_{-n} = (-1)^{n+1}F_n$, we have 4= 2+1+1= 4.

(2) Assume the formula is true for n =1, 2, 3…k-1, k.

(3) Verify the formula for n= k+1.

Note that

$$M_{k+1} = M_k + M_{k-1} = F_{k-3} + F_{k-1} + F_{k+2} + F_{k-4} + F_{k-2} + F_{k+1}$$

$$\Rightarrow \quad M_k + M_{k-1} = F_{k-4} + F_{k-3} + F_{k-2} + F_{k-1} + F_{k+2} + F_{k+1}$$

$$= \; F_{k-2} + F_k + F_{k+3} = M_{k+1}$$

Hence by Induction, the theorem follows.

Example 1: Take n=6. Note that $M_6 = 28$, $F_3 = 2, F_5 = 5, and F_8 = 21.$

Hence $M_6 = F_3 + F_5 + F_8$

Corollary 1.

$M_n = L_n + 2F_{n-1}$.

Proof: BY **Theorem 2**, we have $M_n = F_{n-3} + F_{n-1} + F_{n+2}$. Using the following Fibonacci- Lucas sequences' relations

(1) $F_{n-3} = F_{n-1} - F_{n-2}$

(2) $F_{n+2} = F_{n+1} + F_n$

(3) $L_n = F_{n-1} + F_{n+1}$

We have,

$M_n = F_{n-3} + F_{n-1} + F_{n+2} = F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_n =$
$F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2} = F_{n-1} + F_{n-1} + L_n$.

$\Rightarrow F_{n-3} + F_{n-1} + F_{n+2} = L_n + 2F_{n-1}$.

$\Rightarrow M_n = L_n + 2F_{n-1}$.

Example 2: Take n= 6. Note that $M_6 = 28$, $L_6 = 18 \;\&\; F_5 = 5$.

Thus, M_6 = **28**= L_6**+2**F_5

Theorem 3. (Expressing M in terms of L)

Let M_n **and** L_n be any Mulatu and Lucas Numbers respectively. Then we have:

$$M_n = \frac{7L_n + 2L_{n-2}}{5}$$

Proof: The theorem easily follows from Corollary 1, using the Fibonacci-Lucas relation

$$F_n = \frac{L_{n+1} + L_{n-1}}{5}$$

Example 3: Let n=7. Note that:

$M_7 = 45, L_7 = 29 \,\&\, L_5 = 11.$

We have, $\frac{7(29)+2(11)}{5} = 45 = M_7.$

The Fibonacci and Lucas Numbers have addition formulas. A question may be asked if M has also an addition formula. The answer is positive and produces the following important theorem.

Theorem 4. **<u>The addition formula *for* Mulatu numbers</u>**.

$$M_{n+k} = F_{n-1}M_k + F_nM_{k+1}$$

Proof: By Theorem 2, we have,

$$M_n = F_{n-3} + F_{n-1} + F_{n+2}.$$

Hence it follows that

$$M_{n+k} = F_{n+k-3} + F_{n+k-1} + M_{n+k+2}.$$

Now using the addition formula for Fibonacci numbers given above, it follows that

$$M_{n+k} = (F_{n-1}F_{k-3} + F_n\ F_{k-2}) + (F_{n-1}F_{k-1} + F_nF_k) + (F_{n-1}F_{k+2}\ + F_nF_{k+3})$$

$$= (F_{n-1}F_{k-3}\ + F_{n-1} + \mathrm{F}_{k-1} + F_{n-1}F_{k+2}) + (\mathrm{F}_n\ \mathrm{F}_{k-2} + \mathrm{F}_n\ F_k + F_nF_{k+3})$$

$$= F_{n-1}(F_{k-3}\ + \mathrm{F}_{k-1} + F_{k+2}) +\ F_n(\mathrm{F}_{k-2} + F_k + {}_nF_{k+3})$$

$$= F_{n-1}\ M_k\ + F_n\ M_{k+1}.$$

Hence the theorem is proved.

Example4: Let n= 4 and k=3. Then we have: 45=$M_7 = M_{4+3}$. Note that $F_3 = 2, F_4 = 3, M_3 = 6, \& M_4 = 11$. Hence, $F_3M_4 + F_4M_4$= 2(6)+3(11)=45=M_7

Corollary 2:

$M_{2n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$

Proof: By Theorem 4 we have,

$$M_{2n-1} = M_{n+(n-1)} = F_{n-1}M_{n-1} + F_nM_n$$

$$= F_{n-1}M_{n-1} + F_n(L_n + 2F_{n-1})$$

$$= F_{n-1}M_{n-1} + F_nL_n + 2F_nF_{n-1}$$

$$= F_{n-1}M_{n-1} + F_{2n} + 2F_nF_{n-1}.$$

Now applying *Theorem 4* to M_{n-1}, we have

$$M_{n-1} = M_{(n-1)+0} = F_{n-2}M_0 + F_{n-1}M_1 = 4F_{n-2} + F_{n-1} \text{ and}$$

$$4F_{n-2} + F_{n-1} = 4(F_n - F_{n-1}) + F_{n-1} = -3F_{n-1} + 4F_n.$$

Hence, $M_{2n-1} = F_{2n} + F_{n-1}(-3F_{n-1} + 4F_n) + 2F_nF_{n-1} = F_{2n} - 3F^2{}_{n-1} + 6F_nF_{n-1}$

Example 5. Let n = 5. Then $M_{2n-1} = M_9$. Note that:

(1) $F_{2n} = F_{10} =$ 55 (2) -3 $F^2{}_4$= -3(9) = -27 (3) $6F_5F_4$=6(5) (3) = 90.

We have, 55-27+90=118= M_9.

Corollary 3. The Subtraction formula for **Mulatu** numbers

$$M_{n-k} = 4F_{n-k+1} - 3F_{n-k}$$

Proof: $M_{n-k} = M_{(n-k)+0}$ and hence by Theorem 4, we have

$$M_{n-k} = F_{n-k-1}M_0 + F_{n-k}M_1$$

$$= 4F_{n-k-1} + F_{n-k}$$

$$= 4(F_{n-k-1} + F_{n-k}) - 3F_{n-k}$$

$$= 4F_{n-k+1} - 3F_{n-k}.$$

Example 6.

Let n= 8 and k=3. Then we have: $M_{8-3} = M_5$. Note that $4F_6 = 4(8)$ & $3(F_5) = 15$ Hence, $4F_6 - 3F_5 =$ 32-15=17= M_5.

Lemma1.

$$F_{2n} - M_n F_{n+1} - F_{n+1}F_n = -L^2{}_n$$

Proof: We use the identities listed above to prove the theorem.

Note that $F_{2n} - M_n F_{n+1} - F_{n+1}F_n = F_n L_n - M_n F_{n+1} - F_{n+1}F_n$

$$= F_n(F_{n-1} + F_{n+1}) - F_{n+1}(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(L_n + 2F_{n-1}) - F_{n+1}F_n$$

$$= F_n(F_{n-1} + F_{n+1}) - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(F_{n-1} + F_n + F_{n-1}) - (F_n + F_{n-1})(F_n + F_{n-1} + F_{n-1} + 2F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= F_n(2F_{n-1} + F_n) - (F_n + F_{n-1})(F_n + 4F_{n-1}) - (F_n + F_{n-1})F_n$$

$$= 2F_nF_{n-1} + F^2{}_n - F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F_{n-1}{}^2 - F^2{}_n - F_{n-1}F_n$$

$$= -F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1}$$

$$= -(F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1})$$

$$= -(F_n + 2F_{n-1})^2$$

$$= -(F_n + F_{n-1} + F_{n-1})^2$$

$$= -(F_{n+1} + F_{n-1})^2$$

$$= -L^2{}_n$$

Example 7. Let n =5. Then if follows that:

$F_{2n} - M_n F_{n+1} - F_{n+1}F_n = F_{10} - M_5F_6 - F_6F_5 =$ **55-136-40=** $-121 = -L^2{}_5$

The following result deals with the Double -angle type formula. It is rather an amazingly interesting strong result

Theorem 6. Fundamental identity.

$M_{2n} = M_nL_n + 4(-1)^{n+1}$

Proof: By *Theorem 4*, $M_{2n} = M_{n+n} = F_{n-1}M_n + F_nM_{n+1}$. Again applying Theorem 4, to M_{n+1} and using $L_n = F_{n+1} + F_{n-1}$, we get

$$M_{2n} = F_{n-1}M_n + F_n(F_{n-1}M_1 + F_nM_2)$$

$$= F_{n-1}M_n + F_n(F_{n-1} + 5F_n).$$

$$= F_{n-1}M_n + F_nF_{n-1} + 5F^2{}_n.$$

$$= (L_n - F_{n+1})M_n + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F_{n+1}M_n + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(L_n + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(F_{n+1} + F_{n-1} + 2F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - F_nF_{n-1} - 4F^2{}_{n-1} + F_nF_{n-1} + 5F^2{}_n$$

$$= L_nM_n - F^2{}_n - 4F_nF_{n-1} - 4F^2{}_{n-1} + 5F^2{}_n$$

$$= L_nM_n - (F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1}) + 5F^2{}_n$$

From the proof of Lemma 1, we know that $F^2{}_n + 4F_nF_{n-1} + 4F^2{}_{n-1} = L^2{}_n$.

Hence $M_{2n} = L_nM_n - L^2{}_n + 5F^2{}_n$. Now using that $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$ from above, it easily follows that $M_{2n} = L_nM_n + 4(-1)^{n+1}$.

Example 8:

Let n=5**.** Then if follows that $M_{2n} = M_{10} = 191$. Also we have, $L_nM_n + 4(-1)^{n+1}$ = L_5M_5 +4 $(-1)^6$ **=11(17)+4=191=** M_{10}

Corollary 4: $M_{2n} = L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1}$

Proof: We have

$$\begin{aligned} M_{2n} &= M_nL_n + 4(-1)^{n+1} \\ &= (L_n + 2F_{n-1})L_n + 4(-1)^{n+1} \\ &= L^2{}_n + 2F_{n-1}L_n + 4(-1)^{n+1} \\ &= L^2{}_n + 2F_{n-1}(F_{n+1} + F_{n-1}) + 4(-1)^{n+1} \\ &= L^2{}_n + 2F_{n-1}(F_n + F_{n-1} + F_{n-1}) + 4(-1)^{n+1} \\ &= L^2{}_n + 4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1} \end{aligned}$$

Example 9: Let n =6. Then we have. $M_{12} = 500$. Note that:

$4F^2{}_{n-1} + 2F_nF_{n-1} + 4(-1)^{n+1} + L^2{}_n =$

$4F^2{}_5 + 2F_6F_5 + 4(-1)^7 + L^2{}_6 =$

4(25)+2(8)5-4+324=500= M_{12}

<u>Some Interesting Open Questions</u>.

(1) Are there any more triangular numbers in Mulatu numbers other than 1, 6, 28, and 45? If so, are they finite or infinite?

(2) Are there any more Fermat numbers in Mulatu numbers other than 5 and 17? If so, are they finite or infinite?

(3) Are there any more Fibonacci numbers in Mulatu numbers other than 1 and 5? If so, are they finite or infinite?

(4) Are there any more Lucas numbers in Mulatu numbers other than 1 and 11? If so, are they finite or infinite?

(5) Observe that for n= 1,6,11, 16, and 21 all M, F, and L numbers have the same last digit. Is this pattern finite or infinite?

Noble Dedication: I would like to dedicate this interesting research paper to Dr. Michael Laney, Provost/Vice-President of Academic Affairs of Savannah State University for his great support in recognizing my works and highly encouraging me to continue my research activities. Thank you Dr. Laney for your support and encouragement to develop mathematics nationally as well as internationally. Through your encouragement we are achieving greatness.

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