Epistemological Consequences of the Incompleteness Theorems

Giuseppe Raguní

UCAM - Universidad Católica de Murcia, Avenida Jerónimos 135, Guadalupe 30107, Murcia, Spain - graguni@ucam.edu

After highlighting the cases in which the semantics of a language cannot be mechanically reproduced (in which case it is called *inherent*), the main epistemological consequences of the first incompleteness Theorem for the two fundamental arithmetical theories are shown: the non-mechanizability for the truths of the first-order arithmetic and the peculiarities for the model of the second-order arithmetic. Finally, the common epistemological interpretation of the second incompleteness Theorem is corrected, proposing the new *Metatheorem of undemonstrability of internal consistency*.

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1 Semantics in the Languages

Consider an arbitrary language that, as normally, makes use of a countable[1] number of characters. Combining these characters in certain ways, are formed some fundamental strings that we call *terms* of the language: those collected in a dictionary. When the terms are semantically interpreted, i. e. a certain meaning is assigned to them, we have their distinction in adjectives, nouns, verbs, etc. Then, a proper grammar establishes the rules of formation of sentences. While the terms are finite, the combinations of grammatically allowed terms form an infinite-countable amount of possible sentences.

In a non-trivial language, the meaning associated to each term, and thus to each expression that contains it, is not always unique. The same sentence can enunciate different things, so representing different *propositions*. For example, the same sentence "it is a plain sailing" has a different meaning depending on the circumstances: at board of a ship or in the various cases with figurative sense. How many meanings can be

[1]Finite, as the usual alpha-numeric symbols, or, to generalize, infinite-countable. In this paper we will use either "countable" or "enumerable" with the same meaning, i. e. to indicate that there exists a biunivocal correspondence between the considered set and the set of the natural numbers.

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associated to the same term? That is: how many different propositions, in general, can we get by a single sentence? The answer, for any common semantic language, may be amazing.

Suppose we assign to each term a finite number of well-defined meanings. We could then instruct a computer to consider all the possibilities of interpretation of each term. The computer, to simplify, may assign all the different meanings to an equal number of distinct new terms that it has previously defined. For example, it might define the term "f-sailing" for the figurative use of "sailing" (supposed unique). The machine would then be able, using the grammar rules, to generate all the infinite-countable propositions. In this case we will say that, in the specific language, the meaning has been *eliminated*[2]. More generally we have this case when the different meanings allowed for each term are *effectively* enumerable[3]: even in the case of an infinite-countable amount of meanings, the computer can define an infinite-countable number of new terms and associate only one meaning to each term in order to establish a biunivocal correspondence between sentences and propositions. So, the machine could list all them by combination.

Hence, by definition, we will say that a language is *inherently semantic* (i. e. with a non-eliminable meaning) if it uses at least one term with an amount *not* effectively enumerable of meanings; with the possibility, which we will comment soon, that this quantity is even non-enumerable (or *uncountable*). From the fact that a sentence represents more than one proposition if and only if it contains at least one term differently interpreted, it follows an equivalent condition for the inherent semanticity: a language is inherently semantic if and only if the set of all possible propositions is not effectively enumerable.

Now, the case of uncountable propositions is really what happens in every usual natural language. At first, this feature might surprise or be considered unacceptable: all the meanings that *ever will be assigned* to any settled word are only a countable number. Indeed, a *finite* number! But these meanings cannot be specified once and for all. The fact remains that the *possible* interpretations of the term vary within an infinite collection; moreover, a collection not limited by any prefixed cardinality.

Some classic paradoxes can be interpreted as a confirmation of this property [1]. The *Richard's* one[4], for example, can be interpreted as a meta-proof that the semantic definitions are not countable, i. e. that they are *conceivably* able to define each element of a set with cardinality greater than the enumerable one (and therefore each real number). The proper technique used in a *diagonal Cantorian argument* [2], reveals that the natural language is able to adopt different semantic levels (or contexts), looking "after" (or from the "outside") what was "before" (or "inside") defined; namely, what was previously said by the same language. Identical words used in different contexts have a different meaning and for the number of contexts, including nested, there is no limit.

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[2] Just a concise choice rather than *mechanically reproduced*.

[3] A set is called *effectively* enumerable if there is a machine capable of producing in output all and only its elements.

[4] Where firstly it is admitted that all possible semantic definitions of the real numbers stay in a countable array and, then, one can define, by a *diagonal argument* [2], a real number that is not present in the array.

On the other hand, the *Berry's* paradox [1] clearly shows that a finite amount of symbols, differently interpreted, is able to define an infinite amount of objects. Here the key of the argument is again the use of two different contexts to interpret the verb "define".

2 Inherent Semantics in Arithmetics

Now, on a properly mathematical context, consider the formal first-order arithmetic theory (PA, from *Peano's Arithmetic*). It is possible to prove that its propositions and theorems are effectively enumerable [3], so PA is not inherently semantic. However, from the first incompleteness Theorem follows that *the truths* of PA are not effectively enumerable. In fact, starting by PA, supposed consistent, add as new axioms the class of all its *true* statements. We mean: true in the intuitive (or *standard*) *model* (i. e. consistent interpretation), formed by the spontaneous natural numbers. We have so created a new axiomatic system (PAT) that, by construction, is *syntactically complete*[5]: given any sentence, if it is true is a theorem and if it is false then its negation is a theorem. Now, according to the first incompleteness Theorem we conclude that the axioms of PAT, although enumerable, cannot be effectively enumerable [4-6]. Therefore, for what we said, the truths of the formal first-order arithmetic have to contain inherent semantics.

Equivalently, we can say that in the expression "true statement in the standard model" the term *true* has got an amount not effectively enumerable (although enumerable) of distinct meanings. So, the phrase belongs to an inherently semantic language.

The interesting epistemological consequence of this fact is that no machine can be programmed to get all and only the truths of the Arithmetic. In other words, these truths can only be obtained by the use of not predetermined criteria, so clarifying, interpreting (or inventing!) more and more unpredictable meanings for the concept of truth, in an infinite process. We also can say that the concept of natural number, though spontaneous and primitive, is not *mechanizable*: its nature is inherently semantic.

The PA theory admits an infinite amount of other interpretations besides the standard model. The claim to build an arithmetical theory with the standard interpretation as unique model (briefly: *categorical*), leads to a more general axiomatic system: the (*full*) second-order arithmetic ($FSOA$). Here, the induction principle is extended to *all* the properties of the natural numbers, by a second-order[6] axiom[7] Since these properties, as was proved by Cantor, are uncountable [2], it is admitted that this theory can express a non-enumerable amount of propositions. So, in particular, these are not effectively enumerable: $FSOA$, unlike PA, is an inherently semantic discipline.

[5] An axiomatic theory is called *syntactically complete* if, for any its sentence, it or its negation is a theorem.

[6] For definition, in a *second-order* syntactical expression, the existential quantifiers $\forall$ and $\exists$ can range not only over the variable-elements, but also over variable-properties of the elements of the model. About the dangerous confusions regarding the expressive order, see [7].

[7] See for example [8], where $FSOA$ is called AR.

The categorical nature has, therefore, its price. The (unique) model of *FSOA* reveals all the peculiarities of the standard, intuitive, sight of the natural numbers. Specifically, this unique interpretation has to be able to assign more than a single meaning (indeed a quantity at least $2^{\aleph_0}$) to at least one sentence, before to verify the premises of the theory. So, this model cannot be confused with a conventional uncountable model of a formal, and maintained formal, axiomatic theory. This latter one, considered for example for *PA*, will continue to assign *only one* meaning to each sentence of the system: an uncountable amount of exceeding elements of the universe will have no representation in *PA*. Instead, in the *FSOA* theory the unique model is, in a sense, "dynamical" and a proposition, in general, is not fixed only by its symbolic chain. Semantics is required even to form the propositions!

3 About an improper interpretation of the second incompleteness Theorem

The semantical consequences of the second incompleteness Theorem are often flawed. In reference to a theory that satisfies the same hypothesis of the first incompleteness Theorem, the second one generalizes the undecidability to a class of statements which, interpreted in the *standard model* mean "this system is consistent". Its demonstration, only outlined by Gödel, was published by Hilbert and Bernays in 1939.

The usual interpretation of this Theorem, object of our criticism, is that "every theory that satisfies the hypotheses of the first incompleteness Theorem cannot prove its own consistency". It seems clear, in fact, that the conclusion that a theory cannot prove its own consistency is valid for *all the classical systems*, including non-formal! Moreover, this conclusion does not correspond to the second incompleteness Theorem, but to a new and autonomous metatheorem.

Consider an arbitrary classical axiomatic system. If it is inconsistent, it is deprived of models and therefore of any reasonable interpretation of any statement[8]. Therefore, only the admitting that a given statement of the theory means something, implies agreeing consistency. And indubitably this also applies if the interpretation of the statement is "this system is consistent".

So, if there is no assurance about the consistency of the theory (which, to want to dig deep enough, applies to any mathematical discipline) we cannot be certain on any interpretation of its language. For example, in the case of the usual Geometry, when we prove the pythagorean Theorem, what we really conclude is "if the system supports the Euclidean model (and therefore is consistent), then in every rectangle triangle $c_1{}^2+c_2{}^2=I^2$". Certainly, a deduction with an undeniable epistemological worth, still in the catastrophic possibility of inconsistence.

But now let's see what happens if a certain theorem of a certain theory is interpreted with the meaning: "this system is consistent" in a given interpretation *M*. Similarly, what we can conclude by this theorem is really: "if this system supports the model *M*

[8]More in depth: of any interpretation respecting the principles of contradiction and excluded third.

(and therefore is consistent), then it is consistent". *Something that we already knew and, above all, that does not demonstrate at all the consistency of the system*[9]. Unlike any other statement with a different meaning in M, for this kind of statement we have a peculiar situation: *bothering to prove it within the theory is epistemologically irrelevant in the ambit of the interpretation M.* In more simple words, the statement in question can be a theorem or be undecidable with no difference for the epistemological view. Just it cannot be the denial of a theorem, if M is really a model. So, in any case, *the problem of deducing the consistency of the theory is beyond the reach of the theory itself.* We propose to call *Metatheorem of undemonstrability internal of consistency* this totally general metamathematic conclusion.

Then, the fact that in a particular and hypothetically consistent theory, such a statement is a theorem or is undecidable, is depending on the system and on the specific form of the statement. For theories that satisfy the assumptions of the first one, the second incompleteness Theorem guarantees that "normally" these statements are undecidable. "Normally" just means "in the standard interpretation": in fact the Theorem does not forbid that statements expressing consistency of the system in different models could be theorems[10]. As Lolli says, "it seems that not even a proof shuts discussions" [11]. But in no case this debate can affect the validity of the proposed Metatheorem.

Summarizing, the second incompleteness Theorem identifies another class of essentially undecidable statements for any theory that satisfies the hypothesis of the first one. Whilst the first incompleteness Theorem determines only the Gödel's statement, the second one extends the undecidability to a much broader category of propositions. But, contrary to what is commonly believed, this drastic generalization does not introduce any new and dramatic epistemological concept about the consistency of the system. It doesn't so, even if the Theorem were valid *for every* statement interpretable as "this system is consistent" (which, we reaffirm, seems to be false). Because by it in no case we can conclude that "the system cannot prove its own consistency": this judgment belongs to a different and completely general Metatheorem which seems never have been stated, despite its obviousness and undeniability[11].

Finally we emphasize that the meta-demonstration of the proposed Metatheorem, since refers to any arbitrary classical system, must consist in a purely meta-mathematical reasoning (like that one we have presented): it cannot be formalized.

References

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[9] Just remember that in case of inconsistency we have that every proposition is a theorem!

[10] Indeed, it seems that these statements really exist [9-10].

[11] The consistency of a theory may be demonstrated only outside the same, by another external system. For which, in turn, the problem of consistency arises again. To get out of this endless chain, the "last" conclusion of consistency has to be purely metamathematic. Actually, the most general theory (that demonstrates the consistency of the ordinary mathematic disciplines) is the formal Set Theory and the conclusion of its consistency only consists in a "reasonable conviction".

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[11] Lolli, G. *Op. cit.*, p. 142.