NUMBER OF LEVEL CROSSINGS OF RANDOM ALGEBRAIC POLYNOMIALS

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Abstract—In this paper, we have estimated bounds of the number of level crossings of the random algebraic polynomials $f_n(x,1)=\sum_{k=0}^{n} a_k(t)x^k = 0$ where $a_k(t) \le t, 0 \le t \le 1,$ are dependent random variables assuming real values only and following the normal distribution with mean zero and joint density function $|M|^{1/2}(2\pi)^{-a/s}\exp[(-1/2)\delta' M\delta]$. There exists an integer n_0 and a set E of measure at most $A/(\log n_0 - \log\log\log n_0)$ such that, for each $n>n_0$ and all not belonging to E, the equations (1.1) satisfying the condition (1.2), have at most $\alpha(\log\log n)^2 \log n$ roots where α and A are constants.

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In this paper we use data science to find the number of zeros and number of level crossings of algebraic polynomial using different methods. Data science often uses statistical inferences to predict or analyze trends from data, while statistical inferences uses probability distributions of data. Hence knowing the probability and its applications are important to work effectively on data science problems and we get the number of zeros of the polynomial we consider, is the most approximate to other predecessors.

I. INTRODUCTION

Consider the family of equations

$$f_n(x,1)=\sum_{k=0}^{n} a_k(t)x^k = 0 \quad (1)$$

where $a_k(t) \le t, 0 \le t \le 1,$ are dependent random variables assuming real values only and following the normal distribution with mean zero and joint density function.

$$|M|^{1/2}(2\pi)^{-a/s}\exp[(-1/2)\delta' M\delta] \quad (2)$$

when M^{-1} is the moment matrix with $\sigma_i = 1, \rho_{ij} = \rho, 0 < \rho, i \ne j, i, j = 0,1 \ldots n$ and d' is the transpose of the column vector d.

In this paper we estimate the upper bound of the number of real roots of (1.1). We prove the following theorem.

THEOREM: *There exists an integer n_0 and a set E of measure at most* $A/(\log n_0 - \log\log\log n_0)$ *such that, for each $n>n_0$ and all not belonging to E, the equations*

(1.1) satisfying the condition (1.2), have at most $\alpha(\log\log n)^2 \log n$ *roots where α and A are constants.*

The transformation $x \to \frac{1}{x}$ makes the equation $f_n(x, t)=0$ transformed to

$\sum_{r=0}^{n} a_{n-1}(t)x' = 0$ and $(a0(t).....a_n(t))$

and $(a_n(t), a_{n-1}(t),.....a_0(t)$ have the same joint density function. Therefore number of roots and the measure of the exceptional set in the set [$\infty,-\infty$] are twice the corresponding value can be considered and now show that this upper bound is same as in [0,1].

There are many known asymptotic estimates for the number of real zeros that an algebraic or trigonometric polynomial are expected to have when their coefficients are real random variables. The present paper considers the case where the coefficients are complex. The coefficients are assumed to be independent normally distributed with mean zero. A general formula for the case of any complex non stationary random process is also presented.

Some years ago Kac (1943) gave an asymptotic estimate for the expected number of real zeros of an algebraic polynomial where the coefficients are real independent normally distributed random variables. Later Ibragimov and Maslova (1971) obtained the same asymptotic estimate for a case which included the results due to Kac(1943, 1949), Littlewood and Offord (1939) and others. They considered the case when the coefficients belong to the domain of attraction of normal law. Recently there has been some interesting development of the subject, a general survey of which, together with references may be found in a book by Bharucha-Reid and Sambandham (1986). These generalizations consider different types of polynomials, see for example Dunnage (1966) or study the number of level crossings rather than axis crossings, see Farahmand (1986, 1990). However, they assume the real valued coefficients only. Dunnage (1968) considered a wide distribution for the complex-valued coefficients, however he only obtained an upper limit for the number of real zeros. Indeed, the limitation of this result, being only in the form of an upper bound, is justified. It is easy to see that for the case of complex coefficients there can be no analogue of the asymptotic formula for the expected number of real zeros. To illustrate this point we use the result due to Dunnage (1968).Suppose $\sum_j^n = 0\left(x_j + i\beta_j\right) fj(x)$ has a real root where $f_j(x)$ is in the form of x^j or $\cos^{j\theta}$ and α_j and β_j, $j = 0,1.....n$ are sequences of independent random variables. This implies that the polynomials $\sum_j^n = 0\alpha_j fj(x)$ and $\sum_j^n = 0\beta_j fj(x)$ have a common root and the elimination of $f_j(x)$ leads to the equation

$$\phi(\alpha_0,\alpha_1.........\alpha_n, \beta_0,\beta_1..........\beta_n) = 0.$$

Thus the number of roots in the range $[\infty,-\infty]$ and the measure of the exceptional set are each four times the corresponding estimates for the range [0,1]. Evans has considered the case when the random co-efficients are independent and normal. Our technique of proof is analogous to that of Evans.

$\xi 2.$ We define the circles C_0, C_c, C_m and C_1 as follows. C_0 with centre at z=0 and radius $\frac{1}{2}$, C_e with centre at

$$z=\frac{3}{4}-\frac{\log\log n_0}{2n_0}$$

And of radius

$$\frac{1}{4}-\frac{\log\log n_0}{2n_0}$$

C_m with centre at $z=X^m=1-2^{-m}$ and of radius

$r_m=\frac{1}{2}(1-Xm)=2^{-(m+1)}$ for $m=m_0, m_1 \ldots M$

where

$$m_0=\left[\frac{\log n_0-\log\log\log_n+\log_3}{\log 2}\right]-1$$

and

$$\frac{\log n-\log\log\log_n}{\log 2}-1<M<\frac{\log n-\log\log\log_n}{\log 2}$$

and

$$C1 \text{ with centre at } z=1 \text{ and radius } \frac{\log\log_n}{n}$$

By Jensen's theorem the number of zeros of a regular function $\varphi(z)$ in a circle z_0 and of radius r does not exceed

$$\frac{\log n(M/\varphi(z_0))}{\log(R/r)}$$

where M is the upper bound of $\varphi(z)$ in a concentric circle of radius R. We use this theorem to find the number of zeros of $f_n(z, t)$ in each circle. Summing the number of zeros in each of the circle we get the upper bound of the number of zeros of $f_n(z, t)$ in the circle.

$\xi 3.$ To estimate the upper bound of the number of zeros of $f_n(z, t)$ in the circle C0, we shall use the fact that each $a_k(t)$ has marginal frequent function.

$$\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

Now if $\max|a_v|>(n+1)$ then $|a_v|>(n+1)$ for at least one value of $v\le n$, so that

$$P(\max|a_v|>n+1)\le\sum_{v=0}^{n}P(|a_v|>n+1)$$
$$=(n+1)(2/\pi)^{1/2}\int_{n+1}^{\infty}e^{-t^2/2}dt$$
$$<\left(\tfrac{2}{\pi}\right)^{1/2}e^{-(1/2)(n+1)^2} \qquad (3)$$

Since $|fx(z,t)|\le(n+1)|z|^n\max|a_v|$, in the circle

$$|z|=1+\frac{2\log\log n}{n},$$

We get

$$f_x(x,t)\le\left(1+\frac{2\log\log n}{n}\right)^n(n+1)\max|a_v|$$
$$<(n+1)^2e^{2\log\log n} \qquad (4)$$

Outside a set of measure at most $(2/\pi)^{1/2}e^{-(1/2)(n+1)^2}$ by (3).

$|f_n(0,t)=|a_0||$ and

$$P(|a_0|<(n+1)^{-2})=(2/\pi)^{1/2}\int_0^{(n+1)^{-2}}e^{-n2/t}du<(2/\pi)^{1/2}(n+1)^{-2}.$$

Hence outside a set of measure at most $(2/\pi)^{1/2}(n+1)^{-2}$ we have

$$|f_n(0,t)| = |a_0(t)| \geq (n+1)^{-2}$$

If N_0 denotes the number of zeros of f_n (z, t) in the circle C_0 then Jensen's theorem (J), (4) and (5) we have

$$N_0 < \frac{\log(e^{2\log\log n}(n+1)^4)}{\log 2} = \frac{4\log(n+1)+2\log\log n}{\log 2}$$

Outside a set of measure at most

$$((2/\pi)^{1/2} e^{-(n+1)1/2} + (2/\pi)^{1/2}(n+1)^{-2})$$

Thus for all $n>n_0$, we have

$$N_0 < \frac{4\log(n+1)+2\log\log n}{\log 2}$$

Outside a set of measure at most

$$\sum_{n=n_0+1}^{\infty}(2/\pi)^{1/2}[e^{-(n+1)1/2} + (n+1)^{-1}) < \frac{C}{n_0}$$

Where C is an absolute constant

4. To estimate the upper bound of the number of zeros of f_n (x, t) in the circle C_0 we proceed as follows. The probability that

$$\left|\sum_{v=0}^{n} a_v(t)\left(\frac{3}{4}-\frac{\log\log n_0}{2n_0}\right)^n\right| < (n+1)^{-2}$$

is

$$(2/\pi)^{1/2}\int_0^{(n+1)^{-1}} e^{-n^2/2}du < (2/\pi)^{1/2}(n+1)^{-1}\sigma_n^{-1} \qquad (5)$$

Where

$$\sigma_n^{-1} = (1-\rho)\sum_{v=0}^{n}\left(\frac{3}{4}-\frac{\log\log n_0}{2n_0}\right)^n + \rho\left(\sum_{v=0}^{n}\left(\frac{3}{4}-\frac{\log\log n_0}{2n_0}\right)^n\right)^2$$

$$> (1-\rho)1 - \frac{\exp\left(-2(n+1)\left(\frac{1}{4}+\frac{\log\log n_0}{2n_0}\right)\right)}{1-\left(\frac{3}{4}-\frac{\log\log n_0}{2n_0}\right)^2} \qquad (6)$$

If N_0 denotes the number of zeros of f_n (z, t) in the circle C_0 then Jensen's theorem (J), (4), (5) and (6) we have

$$N_0 < \frac{4\log(n+1)+2\log\log n}{\log 2}$$

Outside a set measure at most

$$\sum_{n=n_0+1}^{\infty}\sqrt{\frac{2}{\pi}}\left(e^{-(n+1)2} + \frac{1}{(n+1)^2\sigma_n}\right) = \frac{C}{n_0^{1/2}}\left[\frac{\log\log n_0}{1-(\log n_0)^{-2}}\right]^{1/2}$$

5. To obtain an upper estimate of the number of zeros of f_n (x, t) in the circle C_m($m=m_0,m_1,\ldots.M$) we need the following lemmas.

LEMMA 1: *Let E be an arbitrary set. Then for complex numbers g, we have*

$$\int_E \log\left|\sum_{v=0}^{n} a_v(t)g_v\right|dt$$

Where

$$< m(E)\log\sigma + m(E)\log\log\left(\frac{C}{m(E)}\right)$$

$$\sigma 2 = (1-\rho)\sum_{v=0}^{\infty}|g_v|^t + \rho\left(\left|\sum_{v=0}^{\infty}|g_v|^t\right|\right)^2$$

PROOF: *Let $g_v=b_v+ic_{vg}$ where b_v and c_v are real. Also let*

$$F = \left\{t : \left|\sum_{v=0}^{\infty}|a_v(t)g_v| \geq A\sigma\right|\right\}$$

$$G = \left\{t : \left|\sum_{v=0}^{\infty}|a_v(t)b_v| \geq A\sigma\right| / 2^{1/2}\right\} \quad \text{No}$$

and

$$H = \left\{t : \left|\sum_{v=0}^{\infty}|a_v(t)c_v| \geq A\sigma / 2^{1/2}\right|\right\}$$

w

$$m(G)=\sigma_n^{-1}(2/\pi)^{1/8}\int_{A_0/2^{1/2}} en^{1/8}\sigma_n^{8}du<\frac{2}{A\pi^{1/2}}en^{1/8}$$

And

$$m(H)\leq\frac{2}{A\pi^{1/2}}en^{1/8}$$

Since

$$F\subset G\cup H \text{ and } m(F)\leq m(G)+m(H)\leq\frac{4}{A\pi^{1/2}}e-d^{1/4}$$

. Following Evans [Lemma] we get the proof of the lemma.

LEMMA 2: *If g_v, v=0,1.....are real and if*

$$G=\left\{t:\left|\sum_{v=0}^{\infty}|a_v(t)g_v|\leq g\sigma\right|\right\}$$

Then m(G)<tQ, where

$$\sigma^2=(1-\rho)\sum_{v=0}^{\infty}g_v^{2}+\rho\left(\sum_{v=0}^{\infty}g_v\right)^2$$

$$\sigma_n^{2}=(1-\rho)\sum_{v=0}^{\infty}g_v^{2}+\rho\left(\sum_{v=0}^{n}g_v\right)^2$$

and

$$Q=(2/\pi)^{1/4}(\sigma/\sigma_n);$$

and if E is any set having no point in common with G then

$$\int_E\log\left|\sum_{v=0}^{n}a_v(t)g_v\right|dt>m(E)\log\sigma-CQm(E)\log\frac{1}{m(E)}$$

PROOF: Following Evans [Lemma2] we get the proof of the lemma.

Let N_m (r, t) denote the number of zeros of $f_x(z,t)$ in the circle with centre x_m and radius r. By Jensen's theorem

$$\int_0^{\delta/1^{m+3}}\frac{N_m(r,t)}{r}dr=\frac{1}{2\pi}\int_{|z-z_m|m-\frac{E}{z^{m+E}}}\log\left|\frac{f_n(z,t)}{f_n(x_m,t)}\right|dz$$

Therefore writing

$\varphi_m(t)$ for $Nm(1/2^{m+1},t)$, we have $\left(2\pi\log\frac{5}{4}\right)^{-1}$

$$\varphi_m(t)\leq\left(2\pi\log\frac{5}{4}\right)^{-1}\int_{|z-z_m|m-\frac{E}{z^{m+E}}}\log\left|\frac{f_n(z,t)}{f_n(x_m,t)}\right|dz$$

and hence we get

$$\varphi_m(t)dt\leq\frac{1}{2\pi\log\frac{5}{4}}\int_0^{2v}d\theta\left\{\int_g\log\left|f_n\left(x_m+\frac{5}{2^{m+3}}e^{i\theta},t\right)\right|dt-\int_E\log\left|f_n(x_m,t)\right|dt\right\}$$

By Lemmas 1 and 2, if E has no point in common with a set G_m of measure at most $Q_m t$ where

$$Q_m=\left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{(1-\rho)\sum_{v=0}^{\infty}x_m^{2v}+\rho\left(\sum_{v=0}^{\infty}x_m^{2v}\right)^2}{(1-\rho)\sum_{v=0}^{\infty}x_m^{2v}+\rho\left(\sum_{v=0}^{\infty}x_m^{v}\right)^2}\right\}^{1/2}$$

We get

$$\varphi_m(t)dt<\frac{m(E)}{2\pi\log\frac{5}{4}}\int_0^{1v}\log V(x_m,\theta)d\theta+CQ_m m(E)\log\frac{1}{m(E)}$$

where

$$V(x_m,\theta)=\frac{(1-\rho)\sum_{v=0}^{\infty}\left|x_m+\frac{5}{2^{m+2}}e^{i\theta}\right|^2+\rho\left(\left|\sum_{v=0}^{\infty}\left(x_m+\frac{5}{2^{m+2}}e^{i\theta}\right)^v\right|\right)^2}{(1-\rho)\sum_{v=0}^{\infty}x_m^{2v}+\rho\left(\sum_{v=0}^{\infty}x_m^{v}\right)^2}$$

Since $|x_m| < 1$ in $V(x_m, 0)$, the second term in both the numerator and denominator is constant. Therefore

$$V(x_m,\theta) < \frac{\left(\sum_{v=0}^{\infty}\left|x_m = \frac{5}{2^{m+2}}e^{i\theta}\right|^{v}\right)^2}{\rho\left(\sum_{v=0}^{\infty} x_m{}^{v}\right)^2}$$

$$< \frac{\left[1-(1-2^{-m})\right]^2}{\rho\left(1-\left(1-\frac{1}{2m}+\frac{5}{2^{m+2}}\right)\right)^2} = \frac{1}{F}\left(\frac{8}{3}\right)^2$$

Hence we obtain

$$\int_E \varphi_m(t)dt < CQ_m m(E)\log\frac{1}{m(E)}$$

If E has no point in common with a set G_m of measure at most Qm/m^2, taking $\varepsilon = m^{-2}$

Consider

$$I = \int_E \frac{\sum_{mt}^{t}\varphi_m(t)}{M(t)\log M(t)}dt,$$

where

$$M(t) \le \Phi(n) = \frac{\log n \log\log\log n}{\log 2}$$

Put

$$E_k = \{t \in E : M(t) - k\}$$

Then

$$E = \bigcup_{k=m_0}^{\Phi(n)} E_k$$

and

$$I = \sum_{k=m_0}^{\Phi(n)} \int_K \frac{\sum_{mt}^{t}\varphi_m(t)}{k\log k}dt, \qquad \sum_1 + \sum_2.$$

where $\sum_1$ contains the terms for which

$$m(E_k) \le m(E)/k^2.$$

First consider $\sum_1$. The function x log x^{-1} is increasing with x for $0 < x < e^{-1}$ and therefore

$$m(E_k)\log\frac{1}{m(E_k)} \le 2m(E)\frac{\log k}{k_2} + \frac{1}{k^2}m(E)\log\frac{1}{m(E)}$$

If $\dfrac{m(E_k)}{k^2} < \dfrac{1}{e}$ or $me^2 > em(E)$.

Now

$$\int_{E_K} \frac{\sum_{mt}^{t}\varphi_m(t)}{k\log k}dt = \frac{1}{k\log k}\sum_{m=m0}^{k}\int_{K}\varphi_m(t)dt,$$
$$< \frac{C}{\log k}(\max \mathrm{Q_n})\mathrm{m(E_k)}\log\frac{1}{\mathrm{m(E_K)}}$$
$$\leq C\left(\frac{P_k}{K^2}m(E) + \frac{P_k}{K^2\log k}m(E)\log\frac{1}{\mathrm{m(E)}}\right)$$

If E_k has no point in common with a set $H_k = \bigcup_{m=m_0}^{k} G_m$ of measure at most $P_k \sum_{m=m_0}^{k}\frac{1}{m^2}$ where $Pk = \max_{m0\leq m\leq k} Q_m$. Now consider $\sum_2$, where $m(E_k)\leq m(E)/k^2$. Then

$$\int_{E_K} \frac{\sum_{mt}^{t}\varphi_m(t)}{k\log k}dt = \frac{1}{k\log k}\sum_{m=m0}^{k}\int_{E_K}\varphi_m(t)dt,$$
$$< \mathrm{C}\frac{P_k}{\log k}\mathrm{m(E_k)}\log\frac{1}{\mathrm{m(E_K)}}$$
$$\leq C\left(P_k m(E_k) + \frac{P_k}{k^2\log k}m(E)\log\frac{1}{\mathrm{m(E)}}\right)$$

If E_k has no point in common with a set H_k . Hence

$$I \leq C\left\{\begin{array}{l}\displaystyle\sum_{\substack{m_0\leq k\leq (n)\\ m(E_k)\leq \frac{m(t)}{k^2}}}\left(\frac{P_k}{K^2}m(E) + \frac{P_k}{K^2\log k}m(E)\log\frac{1}{\mathrm{m(E)}}\right) + \\ \displaystyle\sum_{\substack{m_0\leq k\leq (n)\\ m(E_k)\leq \frac{m(E)}{k^2}}}\left(P_k m(E_k) + \frac{P_k}{\log k}m(E_k)\log\frac{1}{\mathrm{m(E)}}\right)\end{array}\right\}$$

If E has no point in common with a set H of measure at most

$$\left(\frac{1}{m_0}\right)\max_{m0\leq m\leq k} Q_m.$$

Now

$$Q^2{}_m = \left(\frac{2}{\pi}\right)\left(\sigma^2/\sigma_n{}^2\right)$$

$$= \frac{2}{\pi}\frac{(1-\rho)\sum_{v=0}^{\infty} x_m{}^{2v} + \rho\left(\sum_{v=0}^{\infty} x_m{}^v\right)^2}{(1-\rho)\sum_{v=0}^{\infty} x_m{}^{2v} + \rho\left(\sum_{v=0}^{\infty} x_m{}^v\right)^2}$$

$$\leq \frac{4}{\pi}\frac{\left(\sum_{v=0}^{\infty} x_m{}^v\right)^2}{\left(\sum_{v=0}^{\infty} x_m{}^v\right)^2} = \frac{4}{\pi}\frac{1}{\left(1-x_m{}^{n+1}\right)^2}$$

since the second term in both the numerator and denominator is dominant. Therefore

$$Pk < \left(\frac{4}{\pi}\right)^{1/2}\left[1-(1-2^{-\Phi(n)})^{n+1}\right]^{-1}$$

$$\leq \left(\frac{4}{\pi}\right)^{1/2}\left[1-(\log n_0)^{-1}\right]^{-1}$$

$$< \left(\frac{4}{\pi}\right)^{1/2} e2 \text{ if } n_0 > e^{1/\sqrt{2}}$$

Therefore we have

$$I < C_m(E)\log\frac{1}{m(E)}$$

if E has no point in common with a set H of measure at most

$$\frac{e^2}{m_0}\left(\frac{2}{\pi}\right)^{1/2}\frac{C}{m_0}.$$

Thus we obtain that for $n>n_0$ and arbitrary E

$$\int_E \frac{\sum_{m=m_0}^{m(t)} \varphi_m(t)}{M(t)\log M(t)}\,dt < C_m(E)\log\frac{1}{m(E)}$$

If $M(t) \leq \Phi(n)$ and E has no point in common with a set H of measure at most $\frac{C}{m_0}$. Now let

$$F(t) = \sup_{m_0 \leq M \leq \Phi(n)}\left(\frac{\sum_{m=m0}^{m(t)} \varphi_m(t)}{M(t)\log M(t)}\right).$$

II. Conclusions

Hence after solving the theorem and lemmas we have conclude that considering a polynomial (1.1) we have estimate bounds of the number of level crossings of the above random algebraic polynomials where under a given condition with mean zero and joint density function $|M|^{1/2}(2\pi)^{-a/s}\exp[(-1/2)\delta' M\delta]$. There exists an integer n_0 and a set E of measure at most $A/(\log n_0 - \log\log\log n_0)$ such that, for each $n>n_0$ and all not belonging to E, the equations (1.1) satisfying the condition (1.2), have at most $\alpha(\log\log n)^2 \log n$ roots where α and A are constants.

Hence the theorem.

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