Investigation of oscillation on certain dynamic equations

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Abstract: In this paper, some new oscillation criteria for a class of fractional dynamic equations with damping term on time scales are established by use of the properties of fractional calculus and generalized Riccati transformation technique, where the fractional derivative is defined in the sense of the conformable fractional derivative. Oscillation criteria for corresponding dynamic equations on time scales involving integer order derivative are special cases of the present results.

Key–Words: oscillation; fractional dynamic equations; time scales; conformable fractional derivative

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1 Introduction

It is well known that research on qualitative properties of solutions of differential and difference equations is very important in the case their solutions are unknown, such as the stability, existence and so on [1-10]. Recently, Benkhettou etc. developed a conformable fractional calculus theory on arbitrary time scales [11], and established the basic tools for fractional differentiation and fractional integration on time scales. Some properties on the conformable fractional calculus are listed in the following two theorems.

Theorem 1.1. Let $\alpha \in (0,1]$, $a,\ b,\ c \in \mathbb{T}$, $\lambda \in \mathbb{R}$, and $f,\ g$ be two rd-continuous functions. Then the following properties hold:

(i). $\int_a^b [f(t)+g(t)]\Delta^\alpha t = \int_a^b f(t)\Delta^\alpha t + \int_a^b g(t)\Delta^\alpha t.$

(ii). $\int_a^b (\lambda f)(t)\Delta^\alpha t = \lambda \int_a^b f(t)\Delta^\alpha t.$

(iii). $\int_a^b f(t)\Delta^\alpha t = -\int_b^a f(t)\Delta^\alpha t.$

(iv). $\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t.$

(v). $\int_a^a f(t)\Delta^\alpha t = 0.$

(vi). For $|f(t)| \le g(t)$, it holds that $|\int_a^b f(t)\Delta^\alpha t| \le \int_a^b g(t)\Delta^\alpha t.$

(vii). If $f(t) > 0$, then $\int_a^b f(t)\Delta^\alpha t \ge 0.$

Theorem 1.2. Let $\alpha \in (0,1]$, $f,\ g$ be two rd-continuous functions. Then

$$\int_a^b f^{(\alpha)}(t)g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b f(\sigma(t))g^{(\alpha)}(t)\Delta^\alpha t.$$

In this paper, we will consider the following fractional dynamic equation with damping term on time scales of the following form:

$$(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} + p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)} + q(t)x(t) = 0,\ t \in \mathbb{T}_0, \tag{1.1}$$

where $\alpha \in (0,1]$, $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0 = [t_0,\infty)\bigcap \mathbb{T}$, $t_0 > 0$, $a,\ r,\ p,\ q \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

A solution of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory in case all its solutions are oscillatory.

We will establish some new oscillation criteria for Eq. (1.1) by properties of conformable fractional calculus and generalized Riccati transformation technique in Section 2, and present some applications for the established results in Section 3. Some conclusions are presented in Section 4. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = (0,\infty)$, while $\mathbb{Z}$ denotes the set of integers. $t_i \in \mathbb{T}$, $[t_i,\infty)_{\mathbb{T}} = [t_i,\infty)\bigcap\mathbb{T}$, $i = 0,1,...,5$. For the sake of convenience, denote $\delta_1(t,t_i) = \int_{t_i}^{t} \frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s$, where $\widetilde{p}(t) = t^{\alpha-1}p(t)$.

2 Main Results

Theorem 2.1. Assume (2.1)-(2.3) hold:

$$\int_{t_0}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s = \infty, \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta^\alpha s = \infty, \tag{2.2}$$

$$\lim_{t\to\infty}\sup \int_{t_0}^{t}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi = \infty, \tag{2.3}$$

$-\frac{\widetilde{p}}{a} \in \mathfrak{R}_+$, ϕ, φ are two given nonnegative functions on $\mathbb{T}$, and for all sufficiently large t_1, there exists $t_2 > t_1$ such that

$$\lim_{t\to\infty}\sup\{\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \phi(s)[a(s)\varphi(s)]^{(\alpha)} + \frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}$$
$$-\frac{[\phi^{(\alpha)}(s)r(s) + 2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^\alpha s\} = \infty. \tag{2.4}$$

Then every solution of Eq. (1.1) is oscillatory or tends to zero.

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, assume $x(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, for some sufficiently large t_1. By [12, Theorem 2.1 (ii)] it holds either $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ for some sufficiently large $t_2 > t_1$ or $\lim_{t\to\infty} x(t) = 0$.

Now we consider the case $x^{(\alpha)}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. To this end, we define the generalized Riccati function:

$$\omega(t) = \phi(t)a(t)[\frac{(r(t)x^{(\alpha)}(t))^{(\alpha)}}{x(t)e_{-\frac{\widetilde{p}}{a}}(t,t_0)} + \varphi(t)].$$

Then by [12, Theorem 2.1 (i)] one has $\omega(t) \geq 0$, and by [12, Theorem 1.12 (ii)] and [12, Theorem 1.11] one can deduce that

$$\omega^{(\alpha)}(t) = \frac{\phi(t)}{x(t)}[\frac{a(t)(r(t)x^{(\alpha)}(t))^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)}]^{(\alpha)} + [\frac{\phi(t)}{x(t)}]^{(\alpha)}\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}$$
$$+\phi(t)[a(t)\varphi(t)]^{(\alpha)} + \phi^{(\alpha)}(t)a(\sigma(t))\varphi(\sigma(t))$$
$$= \frac{\phi(t)}{x(t)}[\frac{e_{-\frac{\widetilde{p}}{a}}(t,t_0)(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)} - (e_{-\frac{\widetilde{p}}{a}}(t,t_0))^{(\alpha)}a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{\widetilde{p}}{a}}(t,t_0)e_{-\frac{\widetilde{p}}{a}}(\sigma(t),t_0)}]$$

$$
\begin{aligned}
&+[\frac{x(t)\phi^{(\alpha)}(t)-x^{(\alpha)}(t)\phi(t)}{x(t)x(\sigma(t))}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}+\phi^{(\alpha)}(t)a(\sigma(t))\varphi(\sigma(t))\\
&=\frac{\phi(t)}{x(t)}[\frac{(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}+p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))\\
&-[\frac{\phi(t)x^{(\alpha)}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}\\
&=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))-[\frac{\phi(t)x^{(\alpha)}(t)}{x(t)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}.
\end{aligned}
$$

From [12, Theorem 2.2] one furthermore has

$$
\begin{aligned}
&\omega^{(\alpha)}(t)\leq -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))\\
&-(\frac{\phi(t)}{x(t)})\frac{\delta_1(t,t_2)}{r(t)}[\frac{a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}}{e_{-\frac{p}{a}}(t,t_0)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}\\
&\leq -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))\\
&-(\frac{\phi(t)}{x(\sigma(t))})\frac{\delta_1(t,t_2)}{r(t)}[\frac{a(\sigma(t))[r(\sigma(t))x^{(\alpha)}(\sigma(t))]^{(\alpha)}}{e_{-\frac{p}{a}}(\sigma(t),t_0)}]\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}\\
&=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))-[\frac{\phi(t)\delta_1(t,t_2)}{r(t)}][\frac{a(\sigma(t))(r(\sigma(t))x^{(\alpha)}(\sigma(t)))^{(\alpha)}}{x(\sigma(t))e_{-\frac{p}{a}}(\sigma(t),t_0)}]^2+\phi(t)[a(t)\varphi(t)]^{(\alpha)}\\
&=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}\omega(\sigma(t))-[\frac{\phi(t)\delta_1(t,t_2)}{r(t)}][\frac{\omega(\sigma(t))}{\phi(\sigma(t))}-a(\sigma(t))\varphi(\sigma(t))]^2+\phi(t)[a(t)\varphi(t)]^{(\alpha)}\\
&=-q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}-\frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}\\
&+[\frac{\phi^{(\alpha)}(t)}{\phi(\sigma(t))}+2\frac{\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))}{r(t)\phi(\sigma(t))}]\omega(\sigma(t))-\frac{\phi(t)\delta_1(t,t_2)}{r(t)\phi^2(\sigma(t))}\omega^2(\sigma(t))\\
&\leq -q(t)\frac{\phi(t)}{e_{-\frac{p}{a}}(\sigma(t),t_0)}+\phi(t)[a(t)\varphi(t)]^{(\alpha)}-\frac{\phi(t)\delta_1(t,t_2)a^2(\sigma(t))\varphi^2(\sigma(t))}{r(t)}\\
&+\frac{[\phi^{(\alpha)}(t)r(t)+2\phi(t)\delta_1(t,t_2)a(\sigma(t))\varphi(\sigma(t))]^2}{4r(t)\phi(t)\delta_1(t,t_2)}.
\end{aligned}
\tag{2.5}
$$

Substituting t with s in (2.5), fulfilling α-fractional integral for (2.5) with respect to s from t_2 to t yields

$$
\begin{aligned}
&\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{-\frac{p}{a}}(\sigma(s),t_0)}-\phi(s)[a(s)\varphi(s)]^{(\alpha)}+\frac{\phi(s)\delta_1(s,t_2)a^2(\sigma(s))\varphi^2(\sigma(s))}{r(s)}\\
&-\frac{[\phi^{(\alpha)}(s)r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\Delta^{\alpha}s\leq\omega(t_2)-\omega(t)\leq\omega(t_2),
\end{aligned}
$$

which contradicts the condition (2.4), and thus the proof is completed.

Corollary 2.2. in the case $\mathbb{T}=\mathbb{R}$, if we assume that

$$\int_{t_0}^{\infty}\frac{e_{-\frac{p}{a}}(s,t_0)}{a(s)}s^{\alpha-1}ds=\infty, \tag{2.6}$$

$$\int_{t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}ds=\infty, \tag{2.7}$$

$$\int_{t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}ds)\tau^{\alpha-1}d\tau]d\xi=\infty, \tag{2.8}$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\int_{t_2}^{t}\{q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}-\phi(s)s^{1-\alpha}[a(s)\varphi(s)]'+\frac{\phi(s)\delta_1(s,t_2)a^2(s)\varphi^2(s)}{r(s)}$$
$$-\frac{[s^{1-\alpha}\phi'(s)r(s)+2\phi(s)\delta_1(s,t_2)a(s)\varphi(s)]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}s^{\alpha-1}ds\}=\infty, \tag{2.9}$$

where ϕ, φ are two given nonnegative functions on $\mathbb{R}$, then every solution of Eq. (1.1) is oscillatory or tends to zero.

Corollary 2.3. Let $\mathbb{T}=\mathbb{Z}$ and $-\frac{\widetilde{p}}{a}\in\mathfrak{R}_+$. Assume that

$$\sum_{s=t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}=\infty, \tag{2.10}$$

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}=\infty, \tag{2.11}$$

$$\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)})]=\infty, \tag{2.12}$$

and for all sufficiently large t_1, there exists t_2 such that

$$\lim_{t\to\infty}\sup\{\sum_{s=t_2}^{t-1}\{q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}-\phi(s)s^{1-\alpha}[a(s+1)\varphi(s+1)-a(s)\varphi(s)]+\frac{\phi(s)\delta_1(s,t_2)a^2(s+1)\varphi^2(s+1)}{r(s)}$$
$$-\frac{[s^{1-\alpha}(\phi(s+1)-\phi(s))r(s)+2\phi(s)\delta_1(s,t_2)a(\sigma(s))\varphi(\sigma(s))]^2}{4r(s)\phi(s)\delta_1(s,t_2)}\}\}=\infty, \tag{2.13}$$

where ϕ, φ are two given nonnegative functions on $\mathbb{Z}$. Then every solution of Eq. (1.1) is oscillatory or tends to zero.

3 Conclusions

We have presented some oscillation criteria for a class of fractional dynamic equation with damping term on time scales. These results unify continuous and discrete analysis as special cases.

References:

[1] Y. Sun, L. Liu and Y. Wu, The existence and uniqueness of positive monotone solutions for a class of nonlinear Schrodinger equations on infinite domains, J. Comput. Appl. Math., 321 (2017) 478-486.

[2] X. Zheng, Y. Shang and X. Peng, Orbital stability of periodic traveling wave solutions to the generalized Zakharov equations, Acta Math. Sci., 37B (4) (2017) 998-1018.

[3] Z. Zhao, Existence of fixed points for some convex operators and applications to multi-point boundary value problems, Appl. Math. Comput., 215 (8) (2009) 2971-2977.

[4] F. Xu, X. Zhang, Y. Wu and L. Liu, Global existence and temporal decay for the 3D compressible Hall-magnetohydrodynamic system, J. Math. Anal. Appl., 438 (1) (2016) 285-310.

[5] J. Liu, Z. Zhao, Multiple solutions for impulsive problems with non-autonomous perturbations, Appl. Math. Lett., 64 (2017) 143-149.

[6] X. Zheng, Y. Shang and X. Peng, Orbital stability of solitary waves of the coupled Klein-Gordon-Zakharov equations, Math. Methods Appl. Sci., 40 (2017) 2623-2633.

[7] Y. Guan, Z. Zhao and X. Lin, On the existence of positive solutions and negative solutions of singular fractional differential equations via global bifurcation techniques, Bound. Value Probl. 2016:141 (2016) 1-18.

[8] L. Ren and J. Xin, Almost global existence for the Neumann problem of quasilinear wave equations outside star-shaped domains in 3D, Electron. J. Differ. Eq., 2017 (312) (2017) 1-22.

[9] L. Guo, L. Liu and Y. Wu, Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions, Nonlinear Analysis: Modelling and Control, 21 (5) (2016) 635-650.

[10] L. Liu, H. Li, C. Liu and Y. Wu, Existence and uniqueness of positive solutions for singular fractional differential systems with coupled integral boundary value problems, J. Nonlinear Sci. Appl., 10 (2017) 243-262.

[11] N. Benkhettou, S. Hassani and D. F.M. Torres, A conformable fractional calculus on arbitrary time scales, J. King Saud Univer. Sci., 28 (2016) 93-98.

[12] B. Zheng, Some results for a class of conformable fractional dynamic equations on time scales, IJRDO-Journal of Educational Research, 4(5) (2019) 55-60.