Ward-like Operator in the comma theory

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Abstract: The construction of the Ward-like operator of the interacting string field theory presented by Witten, in terms of the half-string (comma) oscillator modes, is completed.

1. Introduction

Witten's cubic bosonic open string field theory was formulated in [1] and made more precise in terms of full string oscillators in [2,3]. In [1], Witten suggested that the full string (FS) may be split into two halves leading to the half-string (comma) approach formulated in [4]. In [5] a precise formulation in terms of half-string oscillators was given. Further development of the comma theory was carried out in [7]. In 2001, the work of references [8–11] generated much interest in the comma formulation of Witten's theory of interacting open bosonic strings. Subsequently, interest in the comma theory faded a way and many of the issues in the comma approach raised in [6,7] remain unsolved. In [14], Ward-like ($\mathcal{WL}$) identities were established and used as part of the proof of the equivalence between the bosonic and fermionic ghost realizations of the gauge invariance in Witten's theory. Although K invariance was established in [6], the $\mathcal{WL}$ identities were not discussed in the case of the comma theory. Just as in the case of the full string, the proof of equivalence in the comma theory requires that similar $\mathcal{WL}$ identities hold in the comma theory. The importance of the proof resides in the fact that the physical content of the theory depends on $\mathcal{WL}$ identities. One of the reasons that this problem received little attention has to do with the difficulty of constructing the $\mathcal{WL}$ operator in the comma approach as we shall see shortly. Once we construct the $\mathcal{WL}$ operator, we can complete the proof of equivalence of the comma three-point vertex (coordinate and ghost), and establish the BRST invariance of the full three-point vertex in the bosonized ghost[15], a problem that was briefly discussed in [6].

2. Comma formalism

Here we shall review the comma formalism needed to describe the $\mathcal{WL}$ operator in terns of the half-string oscillator modes. The comma formalism was developed in [5]. Here we briefly consider the bosonic ghost field $\phi(\sigma)$; the matter fields $x^{\mu}(\sigma)$, $\mu = 0, 1, 2, ..., 25$, have similar structure and have been considered in [5].

2.1. Full string ghost field

The bosonized ghost field $\phi(\sigma)$ for the full string has the mode expansion

$$\phi(\sigma) = \phi_0 + \sqrt{2}\sum_{n=1}^{\infty}\phi_n \cos n\sigma \tag{1}$$

where $0 \leq \sigma \leq \pi$. In (1) the modes ϕ_n are related to creation and annihilation operators through

$$\phi_n = \frac{i}{\sqrt{2n}}\left(a_n^{\phi} - a_{-n}^{\phi}\right) \tag{2}$$

$$p_n^{\phi} = -i\frac{\partial}{\partial\phi_n} = \sqrt{\frac{n}{2}}\left(a_n^{\phi} + a_{-n}^{\phi}\right) \tag{3}$$

for $n > 0$ (with $a_{-n}^{\phi} \equiv a_n^{\phi}$) and

$$\phi_0 = \frac{i}{2}\left(a_0^{\phi} - a_0^{\phi\dagger}\right) \tag{4}$$

$$p_0^{\phi} = -i\frac{\partial}{\partial\phi_0} = \left(a_0^{\phi} + a_0^{\phi\dagger}\right) \tag{5}$$

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for $n = 0$.

2.2. Comma ghost field

Here we review the transformation needed to express the full string ghost field in terms of ghost fields living on the left and right halves of the string. Following ref. [5], we break up the full string ghost coordinate $\phi(\sigma)$ satisfying Neumann boundary conditions at the end points of the string $(\sigma = 0, \pi)$ into its left and right halves, according to

$$\varphi^{(L)}(\sigma) = \phi(\sigma) - \phi(\frac{\pi}{2}), \quad 0 \leq \sigma \leq \frac{\pi}{2} \tag{6}$$

$$\varphi^{(R)}(\sigma) = \phi(\pi - \sigma) - \phi(\frac{\pi}{2}), \quad 0 \leq \sigma \leq \frac{\pi}{2} \tag{7}$$

where now the comma ghost fields $\varphi^{(L)}(\sigma)$, $\varphi^{(R)}(\sigma)$ obeys Neumann boundary condition at $\sigma = 0$ and Dirichlet boundary condition at $\sigma = \pi/2$. The mode expansion of the left and right pieces have been considered in [5]

$$\varphi^{(r)}(\sigma) = \sqrt{2}\sum_{n=1}^{\infty} \varphi_n^{(r)} \cos(2n-1)\sigma \tag{8}$$

where $r = 1, 2 = L, R$. The modes $\varphi_n^{(r)}$ in (8) are related to the comma creation and annihilation operators through

$$\varphi_n^{(r)} = \frac{i}{\sqrt{2n-1}}\left(b_n^{\varphi(r)} - b_{-n}^{\varphi(r)}\right) \tag{9}$$

$$p_n^{\varphi(r)} = -i\frac{\partial}{\partial\varphi_n^{(r)}} = \frac{\sqrt{2n-1}}{2}\left(b_n^{\varphi(r)} + b_{-n}^{\varphi(r)}\right) \tag{10}$$

for $n > 0$ (with $b_{-n}^{(r)} \equiv b_n^{(r)\dagger}$). The degrees corresponding to the midpoint ghost coordinate and and its conjugate momenta are given by

$$\varphi_M \equiv \phi(\pi/2)$$

$$= \phi_0 + \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\varphi_n^{(1)} + \varphi_n^{(2)}\right) \tag{11}$$

and

$$\mathfrak{p}_M^{\varphi} = -i\frac{\partial}{\partial\phi(\pi/2)} = p_0^{\phi}, \tag{12}$$

respectively.

The operators $b's$ are related to the usual comma operators $\beta's$ (corresponds in the full string to $\alpha's$) through

$$b_n^{(r)} = \frac{\beta_{2n-1}^{\varphi(r)}}{\sqrt{2n-1}}, \qquad b_{-n}^{s} = \frac{\beta_{2n-1}^{\varphi(r)\dagger}}{\sqrt{2n-1}} \tag{13}$$

The full string ghost operators α_n^{ϕ} are related to the comma ghost operators β_n^{φ} through [7].

$$\alpha_{2n}^{\phi} = \sum_{m=1}^{\infty} M_{2n\ 2m-1}\beta_{2m-1}^{\varphi(+)} + (-1)^n\,\mathfrak{p}_M^{\varphi}, \tag{14}$$

$$\alpha_{2n-1}^{\phi} = \beta_{2n-1}^{\varphi(+)}, \tag{15}$$

for $n > 0$, and

$$\alpha_0^{\phi} = p_0^{\phi} = \mathfrak{p}_M^{\varphi} \tag{16}$$

for $n = 0$. The operators $\beta_n^{(\pm)}$ are defined by

$$\beta_n^{(\pm)} = \frac{1}{\sqrt{2}}\left(\beta_n^{(1)} \pm \beta_n^{(2)}\right) \tag{17}$$

and the change of representation matrix, M, has the form

$$M_{nm} = \frac{n}{m}\,\frac{(-1)^{(n+m+1)/2}}{n-m} \tag{18}$$

2.3. The three vertex in the comma theory

In the comma approach to string theory, the elements of the theory are defined by $\delta - function$ type overlaps

$$V_3^{(\chi+\varphi)HS} = \exp^{i\sum_{j=1}^{3} Q_j^{(\phi)}\phi(\pi/2)}\, V_{3,0}^{(\varphi)HS} V_{3,0}^{(\chi)HS} \tag{19}$$

where

$$V_{3,0}^{(\varphi)HS} = \prod_{j=1}^{3}\prod_{\sigma=0}^{\pi/2}\delta\left[\varphi_j^{(L)}(\sigma) - \varphi_{j-1}^{(R)}(\sigma)\right] \tag{20}$$

and $V_{3,0}^{(\chi)HS}$ refers to the matter part of the vertex [7]. It is to be understood that $j - 1 = 0 \equiv 3$. The factor $Q_j^{(\phi)}$ is the ghost number insertion

at the mid-point which is needed for the $BRST$ invariance of the theory [2,5] and in this case $Q_1^{(\phi)} = Q_2^{(\phi)} = Q_3^{(\phi)} = 1/2$. As we have seen before [2,5], in the Hilbert space of the theory, the $\delta - functions$ overlaps translate into operator overlap equations which determine the precise form of the vertex. The ghost part of the comma vertex has the same structure as the coordinate one apart from the mid-point insertions

$$|V_3\rangle_{HS}^{\chi+\varphi} = e^{\frac{3}{2}i\phi_M} V_{3,0}^{(\varphi)HS} V_{3,0}^{(\chi)HS} \left| 0, N_g = \frac{3}{2} \right\rangle_{HS}^{\chi+\varphi} \tag{21}$$

where

$$\varphi_M \equiv \phi_1\left(\frac{\pi}{2}\right) = \phi_2\left(\frac{\pi}{2}\right) = \phi_3\left(\frac{\pi}{2}\right) \tag{22}$$

and

$$\left| 0, N_g = \frac{3}{2} \right\rangle_{HS}^{\chi+\varphi} = |0\rangle_L^{\chi} |0\rangle_R^{\chi} \times$$

$$\left| 0, N_g = \frac{3}{2} \right\rangle_L^{\varphi} \left| 0, N_g = \frac{3}{2} \right\rangle_R^{\varphi} \tag{23}$$

In the above expression, $|0\rangle_{123} = |0\rangle_1 |0\rangle_2 |0\rangle_3$ and $V_{3,0}^{HS}$ is the exponential of a quadratic form in the creation operators

$$V_{3,0}^{HS} = \exp\left[\frac{1}{2} \frac{\beta_n^{\dagger r\ i}}{\sqrt{n}} H_{nm}^{ij;ss'} \frac{\beta_m^{\dagger s\ j}}{\sqrt{m}} \right] \tag{24}$$

The matrix elements $H_{nm}^{ij;ss'}$ have the simple form

$$H_{nm}^{rs;\ ij} = 2\delta_{nm}\delta^{rR}\delta^{SL}\delta^{i\ j-1} \tag{25}$$

with $i - 1 = 0 \equiv 3$ and the half string operators $\beta' s$ satisfy the usual commutation relations

$$\left[\beta_n^{r\ i}, \beta_m^{\dagger s\ j} \right] = n\delta^{rs}\delta^{ij}\delta_{n,m}, \quad n, m \text{ odd} \tag{26}$$

It is the simplicity of the coupling matrix, H, that makes the comma theory appealing (Compare H to N in ref. [2]).

3. $\mathcal{WL}$ operator in the FS basis

We recall that in ref. [2,3] the bosonic and fermionic realizations of the full string ghost fields have been used to express the ghost part of the string field but no attention was paid to their equivalence. In ref. [2], the authors showed that both ways give rise to a gauge invariant theory for Witten's open bosonic string. However, their equivalence was never established there and the proof of equivalence of the two formulations of the ghost part was subsequently established in ref. [14]. In ref. [14] it was established that operator vertices obey the same equations and are identical. The key to the proof lies in the various identities satisfied by the Neumann coefficients in the definition of the three point vertex. In the proof of equivalence the authors needed to establish K-invariance and $\mathcal{WL}$ identities (L-equations) which are necessary but not sufficient for gauge invariance which hods only in $D = 26$. They also proved $BRST$-invariance and showed that $BRST$-invariance implies both K-invariance and the WL identities. In [6], the K-invariance of the comma theory was established and the $BRST$-invariance was briefly discussed, but the $\mathcal{WL}$ identities were not addressed. Before discussing the $\mathcal{WL}$ identities and $BRST$-invariance in the comma theory one needs, first, to construct the $\mathcal{WL}$ operator in the comma theory. Before we construct the $\mathcal{WL}$ operator in the comma theory, let us briefly review the $\mathcal{WL}$ identities in the full string oscillator formalism of the open bosonic string.

In the bosonic representation of the ghost, the $\mathcal{WL}$ identities for the Witten vertex matter plus ghost [14] are given by

$$W_m^{x+\phi,i} || V_3^{x+\phi} >= 0, \quad m = 1, 2, ...; \ i = 1, 2, 3 \tag{27}$$

where, $W_m^{x+\phi}$, a Ward-like operator, is defined by

$$W_m^{x+\phi,i} = L_m^{x+\phi,i} + \sum_{j=1}^{3} \sum_{n=0}^{\infty} m\widetilde{N}^{ij} L_{-n}^{x+\phi,j} \tag{28}$$

The full Virasoro generators, $L_m^{x+\phi}$ consist of L_m^x

$+L_m^\phi$, where the ghost Virasoro generators are

$$L_m^\phi = \sum_{n=1}^{\infty} \alpha_n^{\phi\dagger} \alpha_{n+m}^\phi + \frac{1}{2} \sum_{n=1}^{m-1} \alpha_{m-n}^\phi \alpha_n^\phi + p_0^\phi \alpha_m^\phi$$

$$-\frac{3}{2} m \alpha_m^\phi, \quad m > 0 \tag{29}$$

and

$$L_0^\phi = \sum_{n=1}^{\infty} \alpha_n^{\phi\dagger} \alpha_n^\phi + \frac{1}{2} \left(p_0^\phi\right)^2 - \frac{1}{8} \tag{30}$$

Notice that L_m^ϕ differs from the coordinate piece L_m^x only in the extra term, linear in α_m^ϕ, arising because of the extra term in the energy-momentum tensor of $\phi(\sigma)$

$$T_\pm = \frac{1}{2\pi} \left[(\partial_\pm \phi)^2 - \frac{3}{2} \partial_\pm^2 \phi \right] \tag{31}$$

It is this extra term that gives rise to $\mathcal{WL}$ identities in 26 dimensions. With this extra term the Virasoro generators for the ghost satisfy

$$\left[L_n^\phi, L_m^\phi\right] = (n-m) L_{n+m}^\phi - (n-m) \delta_{n+m} \tag{32}$$

The constant term in the commutator $\left[L_n^\phi, L_m^\phi\right]$ is missing from the commutator for the matter part $[L_n^x, L_m^x]$. Likewise the zero mode L_0^ϕ differs from the matter part L_0^x in the constant $-1/8$. This extra term is needed since the vacuum ghost number has the value $p^\phi = 1/2$ and therefore $L_0^\phi = 0$ as desired.

3.1. $\mathcal{WL}$ identities in the comma basis

In the bosonization of the fermionic coordinates for the comma string, using the standard procedure (see Ref. 1 in [3]), it is not obvious that all ingredients of the comma theory employing the comma bosonic fields $\varphi^L(\sigma)$ and $\varphi^R(\sigma)$ are equivalent to those constructed using the original comma fermi fields $c^L(\sigma)$, $c^R(\sigma)$, $b^L(\sigma)$ and $b^R(\sigma)$ appearing in [5]. It has been claimed in [6], that the ghost bosonic vertices in the comma operator formulation obey the same overlap equations as the fermionic vertices and are identical. However a rigorous proof was not presented in [5]. The authors failed to establish that the comma ghost (plus matter) vertex in the bosonic realization of the ghosts satisfy the same $\mathcal{WL}$ identities obeyed by the comma ghost (plus matter) vertex in the fermionic realization of the ghosts. To complete the proof of equivalence between the two realization of the comma ghost vertex, we need to establish the $\mathcal{WL}$ identities utilizing the bosonic representation of the comma ghost fields as well. But to do that one first needs the explicit form of the $\mathcal{WL}$ operator in the comma formalism. Finding the explicit form of the $\mathcal{WL}$ operator in the comma formalism will be the subject of this paper. In a sequel to this paper we will prove that $\mathcal{WL}$ identities do indeed hold in the comma theory of the open bosonic string and use them to establish K-invariance [15]. However, as it is the case in the full string, a direct proof of the $BRST$-invariance that follow from the $\mathcal{WL}$ identities is not possible due to the presence of the $(1/2) L^{\phi,i}$ term in the $BRST$ charge. In [15], we will see that the $\mathcal{WL}$ identities and the K-invariance are necessary but not sufficient conditions for gauge invariance. On the other hand we will see that $BRST$-invariance implies both the $\mathcal{WL}$ identities and the K-invariance [15].

To prove the equivalence between the two ghost realizations in the comma theory we need to show that a similar identity to the one in (27) holds in the comma basis, that is, we need to show that

$$\left[\mathcal{W}_m^{\chi+\varphi,i}(L) + \mathcal{W}_m^{\chi+\varphi,i}(R)\right] | (V_3)_{HS}^{\chi+\varphi} >= 0 \tag{33}$$

holds for $m \geq 0$ and $i = 1, 2, 3$. The operators $W(L)$ and $W(R)$ refer to the left and right parts of the $\mathcal{WL}$ operators, respectively, and are given by

$$\mathcal{W}_m^{\chi(\varphi),i}(L) = \ell_m^{(L)\chi(\varphi),i} + \sum_{j=1}^{3} \sum_{n=0}^{\infty} m \widetilde{N}_{mn}^{ij} \ell_{-n}^{(L)\chi(\varphi),j} \tag{34}$$

and

$$\mathcal{W}_m^{\chi(\varphi),i}(R) = \ell_m^{(R)\chi(\varphi),i} + \sum_{j=1}^{3} \sum_{n=0}^{\infty} m \widetilde{N}_{mn}^{ij} \ell_{-n}^{(R)\chi(\varphi),j}, \tag{35}$$

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for $m \geq 0$, respectively. The operators $\ell_{-n}^{(L)}$ and $\ell_{-n}^{(R)}$ are the comma Virasoro generators corresponding to the left and right parts of the string. One way to obtain the comma ghost Virasoro generators is to start with the full string Virasoro generators defined as

$$L_m = \frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma}\left[\dot{\phi}(\sigma,\tau)+\phi'(\sigma,\tau)\right]^2\Big|_{\tau=0} \tag{36}$$

To use the change of representation from the full string to the comma string in (6) and (7), we need to reduce the range of integration to $\sigma \in [0,\pi/2]$. Thus

$$\begin{aligned} L_m = \frac{1}{4\pi}\int_0^{\pi/2} d\sigma &\left[e^{im\sigma}\left(\dot{\phi}(\sigma)+\phi(\sigma)'\right)^2\right. \\ &\left.+e^{-im\sigma}\left(\dot{\phi}(\sigma)-\phi'(\sigma)\right)^2\right] \\ +\frac{(-1)^m}{4\pi}\int_0^{\pi/2} d\sigma &\left[e^{im\sigma}\left(\dot{\phi}(\pi-\sigma)+\phi(\pi-\sigma)'\right)^2\right. \\ &\left.+e^{-im\sigma}\left(\dot{\phi}(\pi-\sigma)-\phi'(\pi-\sigma)\right)^2\right] \end{aligned} \tag{37}$$

Consistency with the definition of the comma string in (6) and (7), requires that we identify left comma Virasoro generators by

$$\begin{aligned} \ell_m^{(L)} = \frac{1}{4\pi}\int_0^{\pi/2} &\left[e^{im\sigma}\left(\dot{\phi}(\sigma)+\phi(\sigma)'\right)^2\right. \\ &\left.+e^{-im\sigma}\left(\dot{\phi}(\sigma)-\phi'(\sigma)\right)^2\right] d\sigma \end{aligned} \tag{38}$$

and identify right comma Virasoro generators by

$$\begin{aligned} \ell_m^{(R)} = \frac{(-1)^m}{4\pi}\int_0^{\pi/2} &\left[e^{im\sigma}\left(\dot{\phi}(\pi-\sigma)+\phi(\pi-\sigma)'\right)^2\right. \\ &\left.+e^{-im\sigma}\left(\dot{\phi}(\pi-\sigma)-\phi'(\pi-\sigma)\right)^2\right] d\sigma \end{aligned} \tag{39}$$

The full Virasoro generators do not factorize completely into left and right parts. This is because of the midpoint in the definition of the comma ghost coordinates in (6) and (7) since $\varphi^{(L)}(\pi/2) = \varphi^{(R)}(\pi/2) = 0$. One can avoid this problem by execluding the midpoint $\phi(\pi/2)$ from the definition of the half-string ghost coordinates in (6) and (7). If one excludes the midpoint from the definition of the half string ghost coordinates the Virasoro generators will separate into left and right Virasoro generators with the constraint $\lim_{\sigma\to\pi/2}\varphi^{(L)}(\sigma) = \lim_{\sigma\to\pi/2}\varphi^{(R)}(\sigma)$. However, we need not worry about that here. Thus using the definitions in (38) and (39), equation (37) can be written in the form

$$L_m^i = \ell_m^{(L),i} + \ell_m^{(R),i} \tag{40}$$

Henceforth, when the Virasoro generators are expressed in terms of the full string modes we shall write L_m and when they are expressed in terms of the comma modes we shall write $\ell_m = \ell_m^{(L)} + \ell_m^{(R)}$.

The full string vacuum is related to the comma vacua through the relation

$$||0\rangle = \exp\left(-\frac{1}{2}\frac{\beta_{2n-1}^{(+)\dagger}}{\sqrt{2n-1}}\Phi_{mn}\frac{\beta_{2m-1}^{(+)\dagger}}{\sqrt{2m-1}}\right)|0\rangle_L\,|0\rangle_R \tag{41}$$

where $|0\rangle_{L(R)}$ correspond to the comma left (right) vacuum. The operators $\beta^{(+)\prime}s$ are defined by

$$\beta_{2n-1}^{(+)} = \frac{1}{\sqrt{2}}\left(\beta_{2n-1}^{(L)} + \beta_{2n-1}^{(R)}\right) \tag{42}$$

and the matrix Φ has been defined in reference [5]

$$\Phi_{mn} = \frac{\sqrt{2m-1}\sqrt{2n-1}}{2(m+n-1)}\begin{pmatrix} -\frac{1}{2} \\ m-1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ n-1 \end{pmatrix} \tag{43}$$

3.2. The comma WL operator

To construct the comma $\mathcal{WL}$ operator, we first need to derive the Virasoro generators in the comma formalism of string theory. The matter part has been constructed in [5]; hence here we

shall focus on the ghost part of the Virasoro generators.

When quantizing a theory one has to deal with normal ordering ambiguities. In the full string theory the vacuum state $||0>$ is related to the left and right vacuum states $|0>_L$and $|0>_R$ through (41). In the full string formalism [5] one is interested in normal ordering with respect to the full string vacuum (41) but in the comma theory formulated in [4], the relevant normal ordering would be with respect to the comma left and right vacuum states $|0>_L$and $|0>_R$. However, the two criteria are linked through the change of representation relationships.

When ordering operators, we need to regularize the divergent quantities resulting from normal ordering. One way of doing that is based on the so called "heat kernel regularization," in which one replaces the operators $\alpha_n's$ and $\beta_n's$ by

$$\alpha_n^\epsilon = e^{-\left[\frac{n+1}{2}\right]\epsilon}\alpha_n, \qquad \epsilon > 0 \tag{44}$$

$$\beta_n^\epsilon = e_n^{-\left[\frac{n+1}{2}\right]\epsilon}\beta_n, \qquad \epsilon > 0 \tag{45}$$

so that a given operator in any of the representations can be obtained by a limiting procedure for infinitesimally small ϵ. Thus for a given operator O we write

$$O\left(\alpha_n\right) = \lim_{\epsilon\to 0} O\left(\alpha_n^\epsilon\right)$$

$$= \lim_{\epsilon\longrightarrow 0} O\left(e^{-\left[\frac{n+1}{2}\right]\epsilon}\alpha_n\right) \tag{46}$$

for the full string oscillators and

$$O\left(\beta_n^{(r)}\right) = \lim_{\epsilon\to 0} O\left(\beta_n^{(r)\epsilon}\right)$$

$$= \lim_{\epsilon\longrightarrow 0} O\left(e_n^{-\left[\frac{n+1}{2}\right]\epsilon}\beta_n^{(r)}\right) \tag{47}$$

for the comma oscillators $(r = L, R)$. Using this prescription it is now possible to find the relationship between the two normal orderings. For example,

$$:: \alpha_n^\epsilon\alpha_m^\epsilon ::=: \alpha_n^\epsilon\alpha_m^\epsilon : + \frac{1+(-1)^n}{2}$$

$$\times\frac{1+(-1)^m}{2}\left[K_{nm} - \delta_{N+m,0}\theta\left(m\right)\right] \tag{48}$$

where in the above identity :: :: and :: indicate normal ordering with respect to the full string and comma vacuum states, respectively.

To first order in ϵ, the quantity K_{nm} reads [5],

$$K_{nm} = -\frac{1}{2}\frac{nm}{n+m}\left\{\frac{1}{n}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1+|n|}{2}\right)\right]\right.$$

$$+\frac{1}{m}\left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1+|m|}{2}\right)\right]$$

$$\left.\epsilon\left[\psi\left(\frac{1+|n|}{2}\right) - \psi\left(\frac{1+|m|}{2}\right)\right]\right\} \tag{49}$$

where $\psi\left(x\right)$ is the digamma function.

Although now we can express ℓ_m in the comma basis using the definitions in (38) and (39); it is much easier to obtain ℓ_m with the aid of the change of representations relations between the full string and the comma string and then taking care of any ambiguities that can arise from normal ordering of $\alpha's$ and $\beta's$ operators[2]

$$\ell_m\left(\beta_q\right) = \ell_m^{(L)}\left(\beta_q^{(\pm)}\right) + \ell_m^{(R)}\left(\beta_q^{(\pm)}\right) \equiv$$

$$L_m\left(\alpha_q = \left\{\begin{array}{c} M_{q\ q'}\beta_{q'}^{(+)} + (-1)^{q/2}\wp^M \\ \text{for } q = even,\ \ q' = \text{odd} \\ \alpha_q = \beta_q^{(-)} \text{ for } q = \text{odd} \end{array}\right.\right) \tag{50}$$

where $\beta^{(+)}$ has been introduced before and $\beta^{(-)}$ is defined by

$$\beta_{2n-1}^{(-)} = \frac{1}{\sqrt{2}}\left(\beta_{2n-1}^{(L)} - \beta_{2n-1}^{(R)}\right) \tag{51}$$

For ℓ_{2k-1}, skipping tedious but reasonable amount of algebra, equations (50) and (29) yield

$$:: \ell_{2k-1}^{(L)\varphi,i} + \ell_{2k-1}^{(R)\varphi,i} ::=:: \ell_{2k-1}^{\varphi,i} ::= \lim_{\epsilon\longrightarrow 0^+} :: \ell_{2k-1}^{\epsilon\varphi,i} ::$$

$$= \lim_{\epsilon\to 0}\sum_{q=even}\beta_{2k-1-q}^{\epsilon(-),i}\left[\sum_{q'=odd} M_{q\ q'}\beta_{q'}^{\epsilon(+),i}\right.$$

$$\left. + (-1)^{n/2}\wp_M^{\phi,i}\right] - \frac{3}{2}\left(2k-1\right)\beta_{2k-1}^{\epsilon(-),i} \tag{52}$$

[2]We have checked that both ways give precisely the same form. We chose to follow the second path here simply to save space since the second option is much shorter.

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where

$$\beta_q^{(\pm)\epsilon} = e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_q^{(\pm)} \tag{53}$$

The ℓ_{2k-1}^{φ} for the ghost part differs from the ℓ_{2k-1}^{χ} for the coordinate part only in the extra terms linear in $\beta_{2k-1}^{\epsilon(+)}$. Simplifying the above expression, we get

$$:: \ell_{2k-1}^{(L)\varphi,i} + \ell_{2k-1}^{(R)\varphi,i} ::=:: \ell_{2k-1}^{(\epsilon)\varphi,i} ::$$

$$= \wp_M^{\phi,i} \sum_{q \text{ even}} (-1)^{q/2}\, e^{\left|\frac{2k-q}{2}\right|\epsilon}\beta_{2k-1-q}^{(-)\varphi,i}$$

$$+ \sum_{q=even} \sum_{q'=odd} G_k^{\epsilon}(q,q')\,\beta_{2k-1-q}^{(-)\varphi,i}\,\beta_{q'}^{(+)\varphi,i}$$

$$-\frac{3}{2}(2k-1)\,e^{-|k|\epsilon}\beta_{2k-1}^{(-)\varphi,i}, \quad k>0 \tag{54}$$

where $G_k^{\epsilon}(n,q)$, a function of the parameter ϵ , is defined by

$$G_k^{\epsilon}(q,q') = e^{-\left|\frac{2k-q}{2}\right|\epsilon} M_{q\,q'} e^{-|(q'+1)/2|\epsilon} \tag{55}$$

with q even integer and q' odd integer. For $m = even = 2k > 0$,

$$:: \ell_{2k}^{(L)\varphi,i} + \ell_{2k}^{(R)\varphi,i} ::=:: \ell_{2k}^{\varphi,i} ::= \lim_{\epsilon\to 0} :: \ell_{2k}^{(\epsilon)\varphi,i} :: \tag{56}$$

From the above expression and equations (50) and (29), we obtain the even comma Virasoro generators (execluding $m = 0$)

$$:: \ell_{2k}^{(L)\varphi,i} + \ell_{2k}^{(R)\varphi,i} ::=:: \ell_{2k}^{(\epsilon)\varphi,i} ::= \frac{1}{2}\sum_{q \text{ odd}}^{\infty} e^{-\left|\frac{2k-q-1}{2}\right|\epsilon}$$

$$\times e^{-\left|\frac{q+1}{2}\right|\epsilon}\left[\beta_{2k-q}^{(-)\varphi,i}\beta_q^{(-)\varphi,i} + (-1)^k\,\beta_{2k-q}^{(+)\varphi,i}\beta_q^{(+)\varphi,i}\right]$$

$$+(-1)^k\left\{\left[\mathfrak{P}_{\phi}^{i} - \frac{3}{2}(2k)\right]^2 H_k^{\epsilon} + 2\left[\mathfrak{P}_{\phi}^{i} - \frac{3}{2}(2k)\right]\right.$$

$$\left.\sum_{q \text{ odd}}^{\infty} G_k^{\epsilon}(q)\,\beta_q^{(+)\varphi,i} + \sum_{q,q' \text{ odd}}^{\infty} F_k^{\epsilon}(q,q')\,\beta_q^{(+)\varphi,i}\beta_{q'}^{(+)\varphi,i}\right\} \tag{57}$$

where the quantities, H, G, and F, are given by

$$H_k^{\epsilon} = \sum_{q'' \text{ even}}^{\infty} e^{-\left|\frac{2k-q''}{2}\right|\epsilon} e^{-\left|\frac{q''}{2}\right|\epsilon} \tag{58}$$

$$G_k^{\epsilon}(q) = \sum_{q'' \text{ even}}^{\infty} e^{-\left|\frac{q}{2}\right|\epsilon} e^{-\left|\frac{q''}{2}\right|\epsilon} G_k^{\epsilon}(q,q'') \tag{59}$$

$$F_k^{\epsilon}(q,q') = (-1)^k\,\frac{2}{\pi}\frac{(-1)^{\frac{q+1}{2}}}{q} G_k^{\epsilon}(q')$$

$$+\frac{1}{2}\sum_{q'' \text{ even}}^{\infty}\left[1 - e^{\left|\frac{q''}{2}\right|\epsilon} e^{-\left|\frac{2k-q''}{2}\right|\epsilon}\right]$$

$$\times\frac{(-1)^k}{2k-q''} G_k^{\epsilon}(q'',q)\, G_k^{\epsilon}(2k-q'',q') \tag{60}$$

The comma Virasoro generators obtain by taking the limit in ϵ. To first order in ϵ one obtains

$$:: \ell_{2k}^{(L)\varphi,i} + \ell_{2k}^{(R)\varphi,i} ::=:: \ell_{2k}^{\varphi,i} ::= \lim_{\epsilon\to 0} :: \ell_{2k}^{(\epsilon)\varphi,i} ::$$

$$= \frac{1}{2}\sum_{q \text{ odd}}^{\infty}\left[\beta_{2k-q}^{(-)\varphi,i}\beta_q^{(-)\varphi,i} + (-1)^k\,\beta_{2k-q}^{(+)\varphi,i}\beta_q^{(+)\varphi,i}\right] +$$

$$\lim_{\epsilon\to 0}\frac{(-1)^k}{2\epsilon}\left\{\left[\mathfrak{P}^i - \frac{3}{2}(2k)\right]^2 + 2\left[\mathfrak{P}^i - \frac{3}{2}(2k)\right]\frac{2}{\pi}\sum_{q \text{ odd}}^{\infty}\right.$$

$$\frac{(-1)^{\frac{q+1}{2}}}{q} e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i} + \left(\frac{2}{\pi}\right)^2 \sum_{q,q' \text{ odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}$$

$$\left.\frac{(-1)^{\frac{q'+1}{2}}}{q'} e^{-\left|\frac{q+1}{2}\right|\epsilon} e^{-\left|\frac{q'+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i}\beta_{q'}^{(+)\varphi,i}\right\} \tag{61}$$

Likewise for $\ell_0^{\varphi,i}$ one obtains

$$:: \ell_0^{(L)\varphi,i} + \ell_0^{(R)\varphi,i} ::=:: \ell_0^{\varphi,i} ::$$

$$= \frac{1}{2}\sum_{q \text{ odd}}^{\infty}\left[\beta_{-q}^{(-)\varphi,i}\beta_q^{(-)\varphi,i} + \beta_{-q}^{(+)\varphi,i}\beta_q^{(+)\varphi,i}\right] - \frac{1}{8}$$

$$+\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left\{\left(\mathfrak{P}_{\phi}^{i}\right)^2 - 1 + 2\mathfrak{P}_{\phi}^{i}\frac{2}{\pi}\sum_{q \text{ odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}\right.$$

$$\times e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i} + \left(\frac{2}{\pi}\right)^2\sum_{q,q' \text{ odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}$$

$$\left.\times\frac{(-1)^{\frac{q'+1}{2}}}{q'} e^{-\left|\frac{q+1}{2}\right|\epsilon} e^{-\left|\frac{q'+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i}\beta_{q'}^{(+)\varphi,i}\right\} \tag{62}$$

The extra term in (61), linear in $\beta^{(+)}$, arises because of the presence of the $R\phi$ term in the action of the bosonized ghosts [1]

$$I_\phi = \frac{1}{2\pi}\int d^2\sigma\left(\partial_\lambda\phi\partial^\lambda\phi - 3iR\phi\right) \tag{63}$$

or alternatively because of the extra linear term in the ghost energy-momentum tensor (31). The extra term is needed and must have precisely the coefficient given in (31), so that ϕ (ghost coordinate) can cancel the Virasoro anomaly of the x^μ (matter coordinates) so that the total (matter plus ghost) Fourier components of the energy momentum

$$\ell_m^{(\epsilon)s} = \ell_m^{(\epsilon)\chi,s} + \ell_m^{(\epsilon)\varphi,s} - \frac{9}{8}\delta_{m0} \tag{64}$$

obey the Virasoro algebra

$$[\ell_m^{\epsilon,s}, \ell_n^{\epsilon,s}] = (m-n)\,\ell_{m+n}^{\epsilon,r} \tag{65}$$

which is free of central charge.

Utilizing the relationship between the two normal orderings, we can easily deduce the relationship between the Virasoro generators in both cases

$$:: \ell_{2k-1}^{(\epsilon)\chi,\varphi} ::=: \ell_{2k-1}^{(\epsilon)\chi+\varphi} : -\frac{1}{4\epsilon}\left(\frac{1}{\epsilon} + \log\frac{\epsilon}{2} + 1\right) \tag{66}$$

where we have kept only divergent terms of order ϵ.

Now we are in position to write down the explicit form of the $\mathcal{WL}$ operator in the comma basis. We recall that

$$:: \mathcal{W}_m^{\chi+\varphi,i}(L) + \mathcal{W}_m^{\chi+\varphi,i}(R) ::=:: W_m^{\chi+\varphi,i} ::$$

$$=:: \ell_m^{(L)\chi+\varphi,i} + \ell_m^{(R)\chi+\varphi,i} :: + \sum_{j=1}^{3}\sum_{n=0}^{\infty} m\widetilde{N}_{mn}^{ij} :: \ell_{-n}^{(L)\chi+\varphi,j} + \ell_{-n}^{(R)\chi+\varphi,j} :: \tag{67}$$

with $\ell_{-n} \equiv \ell_n^\dagger$.

Let us here focus on the ghost part since the coordinate part will have the same structure with the absence of terms linear in the comma creation and annihilation operators. For $m = odd = 2k-1 > 0$, equation (67) gives

$$:: \mathcal{W}_{2k-1}^{\varphi,i}(L) + \mathcal{W}_{2k-1}^{\varphi,i}(R) ::=:: W_{2k-1}^{\varphi,i} ::$$

$$::=:: \ell_{2k-1}^{(L)\varphi,i} + \ell_{2k-1}^{(R)\varphi,i} :: + \sum_{j=1}^{3}\sum_{n=0}^{\infty}(2k-1)$$

$$\times\widetilde{N}_{2k-1\ n}^{ij} :: \ell_{-n}^{(L)\varphi,j} + \ell_{-n}^{(R)\varphi,j} :: \tag{68}$$

Substituting (54), (61) and (62) into the above expression, after a lengthy algebra, we obtain

$$:: \mathcal{W}_{2k-1}^{\varphi,i}(L) + W_{2k-1}^{\varphi,i}(R) ::=:: W_{2k-1}^{\varphi,i} ::$$

$$= \wp_M^{\phi,j}\sum_{q\ \text{even}}(-1)^{\frac{q}{2}}\, e^{\left|\frac{2k-q}{2}\right|\epsilon}\beta_{2k-1-q}^{(-)\varphi,j}$$

$$+\sum_{q=even}\ \sum_{q'=odd} G_k^\epsilon(q,q')\,\beta_{2k-1-q}^{(-)\varphi,j}\,\beta_{q'}^{(+)\varphi,j}$$

$$-\frac{3}{2}(2k-1)\,e^{-|k|\epsilon}\beta_{2k-1}^{(-)\varphi,j} + \sum_{j=1}^{3}(2k-1)\,\widetilde{N}_{2k-1\ 0}^{ij}$$

$$\times\left\{\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}\left[\beta_q^{(-)\varphi,j}\beta_{-q}^{(-)\varphi,j} + \beta_q^{(+)\varphi,j}\beta_{-q}^{(+)\varphi,j}\right] - \frac{1}{8}\right.$$

$$+\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left[\left(\mathfrak{P}_\phi^j\right)^2 - 1 + 2\mathfrak{P}_\phi^j\frac{2}{\pi}\sum_{q\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}\right.$$

$$\times e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_{-q}^{(+)\varphi,j} + \left(\frac{2}{\pi}\right)^2\sum_{q,q'\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}$$

$$\left.\left.\times\frac{(-1)^{\frac{q'+1}{2}}}{q'}e^{-\left|\frac{q+1}{2}\right|\epsilon}e^{-\left|(q'+1)/2\right|\epsilon}\beta_{-q'}^{(+)\varphi,j}\beta_{-q}^{(+)\varphi,j}\right]\right\}$$

$$+\sum_{j=1}^{3}\sum_{n=1}^{\infty}(2k-1)\,\widetilde{N}_{2k-1\ 2n}^{ij}\left\{\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}e^{-\left|\frac{2n-q-1}{2}\right|\epsilon}\times\right.$$

$$e^{-\left|\frac{q+1}{2}\right|\epsilon}\left[\beta_{-q}^{(-)\varphi,j}\beta_{-(2n-q)}^{(-)\varphi,j} + (-1)^n\,\beta_{-q}^{(+)\varphi,j}\beta_{-(2n-q)}^{(+)\varphi,j}\right]$$

$$+(-1)^k\left[\left(\mathfrak{P}_\phi^j - \frac{3}{2}(2n)\right)^2 H_n^\epsilon + 2\times\right.$$

$$\left.\left(\mathfrak{P}_\phi^j - \frac{3}{2}(2n)\right)\sum_{q\ \text{odd}}^{\infty}G_n^\epsilon(q)\,\beta_{-q}^{(+)\varphi,j}\right.$$

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$$+\sum_{q,q'\ \text{odd}}^{\infty} F_n^{\epsilon}(q,q')\,\beta_{-q'}^{(+)\varphi,j}\ \beta_{-q}^{(+)\varphi,j}\Bigg]\Bigg\}$$

$$+\sum_{j=1}^{3}\sum_{n=1}^{\infty}(2k-1)\,\widetilde{N}_{2k-1\ 2n-1}^{ij}\Bigg\{\wp_M^{\phi,j}\sum_{q\ \text{even}}(-1)^{q/2}e^{|(2n-q)/2|\epsilon}\beta_{-(2n-1-q)}^{(-)\varphi,j}+\sum_{q=even}\ \sum_{q'=odd}G_n^{\epsilon}(q,q')$$

$$\beta_{-q'}^{(+)\varphi,j}\ \beta_{-(2n-1-q)}^{(-)\varphi,j}-\frac{3}{2}(2n-1)\,e^{-|n|\epsilon}\beta_{-(2n-1)}^{(-)\varphi,j}\Bigg\} \tag{69}$$

Likewise for $m=even=2k>0$, equation (67) yields

$$::\mathcal{W}_{2k}^{\varphi,i}(L)+\mathcal{W}_{2k}^{\varphi,i}(R)::=::W_{2k}^{\varphi,i}::=::\ell_{2k}^{(L)\varphi,i}$$

$$+\ell_{2k}^{(R)\varphi,i}::+\sum_{j=1}^{3}\sum_{n=0}^{\infty}2k\widetilde{N}_{2k\ n}^{ij}::\ell_{-n}^{(L)\varphi,j}+\ell_{-n}^{(R)\varphi,j}:: \tag{70}$$

Substituting equations (54), (61) and (62) into the above expression, after a lengthy algebra, we get

$$::\mathcal{W}_{2k}^{\varphi,i}(L)+\mathcal{W}_{2k}^{\varphi,i}(R)::=::W_{2k}^{\varphi,i}::$$

$$=\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}\left[\beta_{2k-q}^{(-)\varphi,i}\beta_q^{(-)\varphi,i}+(-1)^k\,\beta_{2k-q}^{(+)\varphi,i}\beta_q^{(+)\varphi,i}\right]$$

$$+(-1)^k\lim_{\epsilon\to0}\frac{1}{2\epsilon}\Bigg(\left[\mathrm{P}^i-\frac{3}{2}(2k)\right]^2+2\left[\mathrm{P}^i-\frac{3}{2}(2k)\right]$$

$$\times\frac{2}{\pi}\sum_{q\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i}+\left(\frac{2}{\pi}\right)^2\sum_{q,q'\ \text{odd}}^{\infty}$$

$$\frac{(-1)^{\frac{q+1}{2}}}{q}\frac{(-1)^{\frac{q'+1}{2}}}{q'}e^{-\left|\frac{q+1}{2}\right|\epsilon}e^{-\left|\frac{q'+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i}\beta_{q'}^{(+)\varphi,i}\Bigg)$$

$$+\sum_{j=1}^{3}2k\widetilde{N}_{2k0}^{ij}\Bigg\{\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}\Big[\beta_{-q}^{(-)\varphi,j}\beta_q^{(-)\varphi,j}$$

$$+\beta_{-q}^{(+)\varphi,j}\beta_q^{(+)\varphi,j}\Big]-\frac{1}{8}+\lim_{\epsilon\to0}\frac{1}{2\epsilon}$$

$$\Bigg(\left(\mathrm{P}_\phi^j\right)^2-1+2\mathrm{P}_\phi^j\frac{2}{\pi}\sum_{q\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}$$

$$\times e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_{-q}^{(+)\varphi,j}+\left(\frac{2}{\pi}\right)^2\sum_{q,q'\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}\times$$

$$\frac{(-1)^{\frac{q'+1}{2}}}{q'}e^{-|(\frac{q+1}{2})|\epsilon}e^{-\left|\frac{q'+1}{2}\right|\epsilon}\beta_{-q'}^{(+)\varphi,j}\beta_{-q}^{(+)\varphi,j}\Bigg)\Bigg\}$$

$$+\sum_{j=1}^{3}\sum_{n=1}^{\infty}2k\widetilde{N}_{2k\ 2n}^{ij}\Bigg\{\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}e^{-\left|\frac{2n-q-1}{2}\right|\epsilon}e^{-\left|\frac{q+1}{2}\right|\epsilon}$$

$$\times\left[\beta_{-q}^{(-)\varphi,j}\beta_{-(2n-q)}^{(-)\varphi,j}+(-1)^n\,\beta_{-q}^{(+)\varphi,j}\beta_{-(2n-q)}^{(+)\varphi,j}\right]$$

$$+(-1)^k\Bigg[\left(\mathrm{P}_\phi^j-\frac{3}{2}(2n)\right)^2H_n^{\epsilon}+2\left(\mathrm{P}_\phi^j-\frac{3}{2}(2n)\right)\times$$

$$\sum_{q\ \text{odd}}^{\infty}G_n^{\epsilon}(q)\,\beta_{-q}^{(+)\varphi,j}+\sum_{q,q'\ \text{odd}}^{\infty}F_n^{\epsilon}(q,q')\,\beta_{-q'}^{(+)\varphi,j}\ \beta_{-q}^{(+)\varphi,j}\Bigg]\Bigg\}$$

$$+\sum_{j=1}^{3}\sum_{n=1}^{\infty}2k\widetilde{N}_{2k\ (2n-1)}^{ij}\Bigg\{\wp_M^{\phi,j}\sum_{q\ \text{even}}(-1)^{\frac{q}{2}}\,e^{\left|\frac{2n-q}{2}\right|\epsilon}$$

$$\times\beta_{-(2n-1-q)}^{(-)\varphi,j}+\sum_{q=even}\ \sum_{q'=odd}G_n^{\epsilon}(q,q')\,\beta_{-q'}^{(+)\varphi,j}$$

$$\beta_{-(2n-1-q)}^{(-)\varphi,j}-\frac{3}{2}(2n-1)\,e^{-|n|\epsilon}\beta_{-(2n-1)}^{(-)\varphi,j}\Bigg\} \tag{71}$$

And last for $m=0$, equation (67) yields

$$::\mathcal{W}_0^{\varphi,i}(L)+\mathcal{W}_0^{\varphi,i}(R)::$$

$$=::\mathcal{W}_0^{\chi+\varphi,i}:=::\ell_0^{(L)\varphi,i}+\ell_0^{(R)\varphi,i}::$$

$$=\frac{1}{2}\sum_{q\ \text{odd}}^{\infty}\left[\beta_{-q}^{(-)\varphi,i}\beta_q^{(-)\varphi,i}+\beta_{-q}^{(+)\varphi,i}\beta_q^{(+)\varphi,i}\right]-\frac{1}{8}$$

$$+\lim_{\epsilon\to0}\frac{1}{2\epsilon}\Bigg\{\left(\mathrm{P}_\phi^i\right)^2-1+2\mathrm{P}_\phi^i\frac{2}{\pi}\sum_{q\ \text{odd}}^{\infty}\frac{(-1)^{\frac{q+1}{2}}}{q}$$

$$\times e^{-\left|\frac{q+1}{2}\right|\epsilon}\beta_q^{(+)\varphi,i} + \left(\frac{2}{\pi}\right)^2 \sum_{q,q' \text{ odd}}^{\infty} \frac{(-1)^{\frac{q+1}{2}}}{q} \times$$

$$\frac{(-1)^{\frac{q'+1}{2}}}{q'} e^{-\left|\frac{q+1}{2}\right|\epsilon} e^{-\left|\frac{q'+1}{2}\right|\epsilon} \beta_q^{(+)\varphi,i} \beta_{q'}^{(+)\varphi,i} \Bigg\} \quad (72)$$

The corresponding coordinates parts of the $\mathcal{WL}$ operator are given by the same expressions with the terms linear in $\beta' s$ deleted.

4. Conclusions

We have seen in [5] that the comma vertex can be expressed in terms of bose or fermi fields. Part of the proof of equivalence of the two forms requires finding the $\mathcal{WL}$ operator in the comma theory. We have constructed the $\mathcal{WL}$ operator in the comma formulation of Witten's open bosonic string which was the main obstacle in proving the $\mathcal{WL}$ identities. The proof of the $\mathcal{WL}$ identities and $BRST$- invariance are given in a sequel to this paper [15].

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