Characterizing the Multivariate Exponential Distributions

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Abstract

The vast majority of the multivariate exponential distributions arise in the reliability context one way or another. When we talk of reliability, we have in mind the failure of an item or death of a living organism. We especially think of time elapsing between the equipment being put into service and its failure. In the bivariate or multivariate context, we are concerned with dependencies between two failure times, such as those of two components of an electrical, mechanical, or biological system.

Bivariate and multivariate exponential distributions have served as friendly “alternative arena” for those involved in theoretical and/or applied aspects of multivariate distributions. The volume on the exponential distribution prepared by Balakrishnan and Basu (1995) provides ample testimony to this fact.

In this article, a characterization of the exponential distribution based on the properties of the bivariate exponential is discussed. The result forms a sort of multivariate analogue of the characterization of the bivariate exponential distribution.

Although different forms of bivariate exponential distributions such those of Gumbel (1960), Freund (1961), Marshall and Olkin (1967) and Block and Basu (1974) exist in literature, how far these distributions can be characterized by properties analogous to the results in the bivariate exponential distribution.

At the beginning we present a detailed discussion on bivariate exponential distributions, describing many different forms that have been proposed in the literature, their properties and applications, and inferential issues. Next, we summarize various developments on multivariate exponential distributions. It should be mentioned that although this chapter includes numerous results from the voluminous literature on this topic, it can by no means be regarded as an exhaustive coverage of this active area of research.

Key words and phrases: characterization, multivariate exponential and bivariate exponential distributions,

Bivariate Exponential Distributions

The term bivariate exponential usually refers to bivariate distributions with both marginal distributions being exponential (BEDs). It is mostly the case that these are standard exponential distributions, but location and scale parameter can be easily introduced, if needed, through appropriate linear transformations.

First we mention a simple special case of the bivariate gamma distributions. The special case of α =2, with a reparameterization, yields the joint density function

$$p_{X_1,X_2}(x_1,x_2) = \frac{\theta_1\theta_2}{1-\rho} I_0\left(\frac{2\sqrt{\rho\theta_1\theta_2x_1x_2}}{1-\rho}\right) exp\left(-\frac{\theta_1x_1+\theta_2x_2}{1-\rho}\right), \quad x_1,x_2 > 0$$

Where $I_0(z) = \sum_{j=0}^{\infty}\left(\frac{z}{2j!}\right)^{2j}$ is the well-known modified Bessel function of the first kind of order zero. Note that X_1 and X_2 are mutually independent if and only if ρ =0. This is the so-called Moran-Downton bivariate exponential distribution.

There are now a number of different kinds of bivariate exponential distributions. However, it was only in 1960 that a pioneering paper, specifically devoted to bivariate exponential distributions.

We now list some systems of bivariate exponential distributions, starting with three systems adumbrated by Gumbel (1960).

Gumbel's Type I Bivariate Exponential Distribution

The joint cumulative distribution function is

$$H(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, \qquad x,y \geq 0, 0 \leq \Theta \leq 1.$$

Characterizations

Along with the bivariate Lomax distribution and bivariate finite range distribution, Gumbel's type I bivariate exponential distribution can be characterized through

- Constant product of bivariate mean remaining (residual) lives and hazard rates
- Constant coefficient of variation of bivariate residual lives.

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Estimation Method

By introducing scale parameters to the marginal distributions, the survival function corresponding to above JCD after relabeling ϴ by α becomes

$$\bar{H}(x,y) = exp\left\{-\frac{x}{\theta_1} - \frac{y}{\theta_2} - \frac{\alpha xy}{\theta_1\theta_2}\right\}, \qquad x, y > 0, \theta_1, \theta_2 > 0, 0 < \alpha < 1.$$

Kotz et al. (2003) derived the distributions of $T_1 = \min(X,Y)$ and $T_2 = \max(X,Y)$. In particular, it was shown that

$$E(T_1) = e^{1/\theta}\sqrt{\frac{\pi}{\theta}}\left[1 - \Phi(\sqrt{2/\theta}\right]$$

And that

$$E(T_2) = 2 - e^{1/\theta}\sqrt{\frac{\pi}{\theta}}\left[1 - \Phi(\sqrt{2/\theta}\right]$$

Further, it was shown that $E(T_2)$ is almost linearly increasing in ρ. Franco and Vivo (2006) discussed log-concavity of the extremes. The distribution that has a log-concave density has an increasing likelihood ratio.

Gumbel's Type II Bivariate Exponential Distribution

Gumbel's type II bivariate exponential distribution is simply an F-G-M model with exponential marginals. The density function is given by

$$h(x,y) = e^{-x-y}\{1 + \alpha(2e^{-x} - 1)(2e^{-y} - 1)\}, \qquad |\alpha| < 1.$$

Bilodeau and Kariya (1994) observed that the density functions of both type I and type II are of the form

$$h(x,y) = \gamma_1\gamma_2 g(\gamma_1 x, \gamma_2 y; \theta)e^{-\gamma_x - \gamma_2}$$

Fisher Information

Nagaraja and Abo-Eleneen (2002) derived expressions for the elements of the Fisher information matrix for the three elements of the Gumbel type II bivariate exponential distribution. They observed that the improvement in the efficiency of the maximum likelihood estimate of the mean of X due to availability of the covariate values as well as the knowledge of the nuisance parameters is limited for this distribution.

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Gumberl's Type III Bivariate Exponential Distribution

The joint cumulative distribution function is

$$H(x,y) = 1 - e^{-x} - e^{-y} + exp\left\{-(x^m + y^m)^{1/m}\right\}, \quad x, y \geq 0, m \geq 1.$$

The survival function is

$$\bar{H}(x,y) = exp\left\{-(x^m + y^m)^{1/m}\right\}$$

The corresponding joint density function is

$$h(x,y) = (x^m + y^m)^{-2+(1/m)} x^{m-1} y^{m-1} \left\{(x^m + y^m)^{1/m+m-1}\right\} exp\left\{-(x^m + y^m)^{1/m}\right\},$$
$$x, y \geq 0, m > 1.$$

If m=1 X and Y are mutually independent. Lu and Bhattacharyya (1991) have studied this bivariate distribution in detail and in particular provided several inferential procedures for this model.

Freund's Bivariate Distribution

This distribution is often given the acronym BEE (bivariate exponential extension) because it is not a bivariate exponential distribution in the traditional sense, as the marginals are not exponentials.

Formula of the Joint Density

The joint density function is

$$h(x,y) = \begin{cases} \alpha\acute{\beta} \exp\left[-(\alpha + \beta - \acute{\beta})x - \acute{\beta}y\right] & for\ x \leq y \\ \acute{\alpha}\beta \exp\left[-(\alpha + \beta - \acute{\alpha})y - \acute{\alpha}x\right] & for\ x \geq y \end{cases}$$

where $x, y \geq 0$ and the parameters are all positive

Formula of the Cumulative Distribution Function

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The joint cumulative distribution function is

$$H(x,y) = \begin{cases} \frac{\alpha}{\alpha+\beta-\acute{\beta}} \exp[-(\alpha+\beta-\acute{\beta})x - \acute{\beta}y] + \frac{\beta-\acute{\beta}}{\alpha+\beta-\acute{\beta}} \exp[-(\alpha+\beta)y] & for\ x \leq y \\ \frac{\beta}{\alpha+\beta-\acute{\alpha}} \exp[-(\alpha+\beta-\acute{\alpha})y - \acute{\alpha}x] + \frac{\alpha-\acute{\alpha}}{\alpha+\beta-\acute{\alpha}} \exp[-(\alpha+\beta)x] & for\ x \geq y \end{cases}$$

where $x, y \geq 0$

Univariate Properties

The marginal distributions are not exponential, but they are mixtures of exponentials. Hence, the joint density function is often known as Freund's bivariate exponential extension, or a bivariate exponential mixture distribution, as it is called by Kotz et al. (2000, p. 356). The expression for the marginal density *f(x)* is

$$f(x) = \frac{(\alpha-\acute{\alpha})(\alpha+\beta)}{\alpha+\beta-\acute{\alpha}} e^{-(\alpha+\beta)y} + \frac{\acute{\alpha}\beta}{\alpha+\beta-\acute{\alpha}} e^{-\acute{\alpha}x}$$

Provided $\alpha+\beta \neq \acute{\alpha}$, and naturally a similar expression for *g(y)* holds wih β and $\acute{\beta}$ changed to α and $\acute{\alpha}$, respectively. The special case of $\alpha+\beta = \acute{\alpha}$ gives $f(x) = (\acute{\alpha}\beta x + \alpha)e^{-\acute{\alpha}x}$.

The mean and variance of this distribution are $\frac{\acute{\alpha}+\beta}{\alpha(\acute{\alpha}+\beta)}$ and $\frac{\acute{\alpha}^2+2\alpha\beta+\beta^2}{\acute{\alpha}^2(\alpha+\beta)^2}$ respectively.

Correlation Coefficient

Pearson's correlation coefficient is given by

$$\frac{\acute{\alpha}\acute{\beta} - \alpha\beta}{\sqrt{(\acute{\alpha}^2 + 2\alpha\beta + \beta^2)(\acute{\beta}^2 + 2\alpha\beta + \alpha^2)}}$$

Which is restricted to the range $-\frac{1}{3}$ to 1

Joint Moment Generating Function

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The joint m.g.f is

$$M(s,t) = (\alpha + \beta - s - t)^{-1}[\frac{\acute{\alpha}\beta}{\acute{\alpha} - s} + \frac{\alpha\acute{\beta}}{\acute{\beta} - t}]$$

Marshall and Olkin's Bivariate Exponential Distribution

It is one of the most widely studied bivariate exponential distributions. The acronym BVE is often used in the literature to designate this distribution.

Formula of the Cumulative Distribution Function

The cumulative distribution function is

$$\overline{H}(x,y) = \exp[-\gamma_1 x - \gamma_2 y - \gamma_{12}\max(x,y)], \qquad x,y \geq 0$$

Where all γ's are positive.

Formula of the Joint Density Function

This takes slightly different forms depending on whether x or y is bigger:

$$h(x,y) = \begin{cases} \gamma_2(\gamma_1 + \gamma_{12})\exp[-(\gamma_1 + \gamma_{12})x - \gamma_2 y] & for\ x > y, \\ \gamma_1(\gamma_2 + \gamma_{12})\exp[-\gamma_1 x + (\gamma_2 + \gamma_{12})y] & for\ y > x, \\ Singularity\ along\ the\ diagonal & for\ x = y \end{cases}$$

The amount of probability for the singular part is $\gamma_{12}/(\gamma_1 + \gamma_2 + \gamma_{12})$. The singularity in this case is due to the possibility of X exactly equaling Y. in the reliability context, this corresponds to the simultaneous failure of the two components.

Univariate Properties

Both marginal distributions are exponential.

Conditional Distribution

The conditional density of Y given X=x is

$$h(y\backslash x)=\begin{cases}\dfrac{\gamma_1(\gamma_2+\gamma_{12})}{\gamma_1+\gamma_{12}}e^{-\gamma_2 y-\gamma_{12}(y-x)} & for\ y>x,\\ \gamma_2 e^{-\gamma_2 y}\gamma_1 & for\ y<x.\end{cases}$$

Correlation Coefficients

Pearson's product-moment correlation coefficient is $\gamma_{12}/(\gamma_1+\gamma_2+\gamma_{12})$.

Characterizations

Block (1977) proved that X and Y with exponential marginals have Marshall and Olkin's bivariate exponential distribution if and only if one of the following two equivalent conditions holds:

- min(X, Y) has an exponential distribution,
- X − Y and min(X, Y) are independent.

Other Properties

- The joint moment generating function is

$$M(s,t)=\frac{(\gamma+s+t)(\gamma_1+\gamma_{12})(\gamma_2+\gamma_{12})+\gamma_{12}st}{(\gamma_1+\gamma_{12}-s)(\gamma_2+\gamma_{12}-t)}$$

- Min (X,Y) is exponential and Max (X,Y) has a survival function given by

$$e^{-(\gamma_1+\gamma_{12})x}+e^{-(\gamma_2+\gamma_{12})x}-e^{-(\gamma_1+\gamma_2+\gamma_{12})x}, \qquad x>0$$

- The aging properties of minimum and maximum statistics were discussed by Franco and Vivo (2002), who showed that max(X, Y) is a generalized mixture of three exponential components. The distribution is neither ILR (increasing likelihood ratio) nor DLR (decreasing likelihood ratio). Because the minimum statistic is exponentially distributed, it is therefore both ILR and DLR.
- The distribution is not infinitely divisible except in the degenerate case when γ_1 = 0 (or γ_2 = 0) or when γ_{12} = 0 (in the latter case, X and Y are independent).

Friday and Patil's Bivariate Exponential

Friday and Patil (1977) proposed a distribution that subsumes both Freund's and Marshall and Olkin's distributions with joint survival function

$$\bar{H}(x,y) = \gamma(\bar{H}_A(x,y) + (1-\gamma)\bar{H}_B(x,y)$$

Where $\bar{H}_A$ is the survival function corresponding to Freund's distribution, and $\bar{H}_B$ is the singular distributionexp$[-(\alpha+\beta)\max(x,y)]$. More explicitly, we have

$$\bar{H}(x,y) = \begin{cases} \theta_1 \exp[-(\alpha+\beta-\acute{\beta})x - \beta y] + (1-\theta_1)\exp[-(\acute{\alpha}+\beta)y] & for\ x \le y, \\ \theta_2 \exp[-\alpha\acute{x} - (\alpha+\beta-\acute{\alpha})y] + (1-\theta_2)\exp[-(\alpha+\acute{\beta})x] & for\ x \ge y, \end{cases}$$

Where $\theta_1 = \gamma\alpha(\alpha+\beta-\acute{\beta})^{-1}, \theta_2 = \gamma\beta(\alpha+\beta-\acute{\alpha})^{-1}$, and $0 \le \gamma \le 1$. This distribution is another one that has the lack of memory property. It is sometimes denoted by BEE.

Friday and Patil also showed that only two independent standard exponential variates are needed to generate a pair (X, Y) with their distribution, and thus the same is true for Freund's and Marshall and Olkin's distributions. They then examined the computational efficiency of their scheme.

The model of Platz (1984) is another one that includes both Marshall and Olkin and Freund models and in addition one-out-of-three and two-out-of three systems with identical components.

The maximum is either a generalized mixture of three or fewer exponentials or a mixture of gamma and exponentials.

Raftery's Bivariate Exponential

In its general form, Raftery's (1984,1985) scheme of obtaining a bivariate distribution with exponential marginal is given by

$$\begin{cases} X = (1-p_{10}-p_{11})U + I_1 W \\ Y = (1-p_{01}-p_{11})V + I_2 W \end{cases}$$

Where U,V,W are independent and exponentially distributed, and in addition, they are independent of I_i. $I_1$and I_2 are each either 0 or 1, with probabilities as set out below

	$I_2=0$	$I_2=1$
$I_1=0$	P_{00}	P_{01}
$I_1=1$	P_{10}	P_{11}

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Raftery showed the correlation to be $2p_{11}-(p_{01}+p_{11})(p_{10}+p_{11})$. There is also an extension of the model to permit negative correlation. Raftery then paid special attention to the following cases

First Special Case

This set $p_{01}=0,\ p_{10}=1-p_{11}$, so that

$$\begin{cases} X = W \\ Y = (1-p_{11})V + I_2 W \end{cases}$$

And the distribution in this case is a mixture of independence and weighted linear combination.

Formula of the Cumulative Distribution Function

The joint survival function that corresponds to first special case with $\delta = p_{11}$ is

$$\bar{H}(x,y)\begin{cases} e^{-x} + \dfrac{1-\delta}{1+\delta} e^{-x/(1-\delta)}\left\{e^{y^{\delta/(1-\delta)}} - e^{-y/(1-\delta)}\right\} & for\ x \geq y, \\ e^{-y} + \dfrac{1-\delta}{1+\delta} e^{-y/(1-\delta)}\left\{e^{x^{\delta/(1-\delta)}} - e^{-x/(1-\delta)}\right\} & for\ x \leq y, \end{cases}$$

Moran-Downton Bivariate Exponential Distribution

This bivariate exponential distribution was first introduced by Moran (1967) and then popularized by Downton (1970). In fact, it is a special case of Kibble's bivariate gamma distribution. Many authors simply call it Downton's bivariate exponential distribution.

Formula of the Joint Density

The joint density function is

$$h(x,y) = \frac{1}{1-\rho}\exp\left[-\frac{x+y}{1-\rho}\right] I_0\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right), \qquad x, y \geq 0$$

where I_0 is the modified Bessel function of the first kind of order zero.

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Formula of the Cumulative Distribution Function

Expressed as an infinite series, the joint cumulative distribution function is

$$H(x,y) = (1-e^{-x})(1-e^{-y}) + \sum_{j=0}^{\infty} \frac{\rho^{j+1}}{(j+1)^2} L_j^{(1)}(x) L_j^{(1)}(y) xye^{-(x,y)}$$

for $x, y \geq 0$, where the $L_j^{(1)}$ are laguerre polynomials.

Univariate Properties

Both marginal distributions are exponential.

Correlation Coefficients

The value ρ is in fact Pearson's product-moment correlation. As to the estimation of ρ, Al-Saadi and Young (1980) obtained the maximum likelihood estimator, the method of moments estimator, the sample correlation estimator, and the two bias-reduction estimators.

Balakrishnan and Ng (2001) proposed two modified bias-reduction estimators, $\tilde{\rho}_5$ and $\tilde{\rho}_6$, and their jackknifed versions, $\tilde{\rho}_{5,J}$ and $\tilde{\rho}_{6,J}$, respectively. They carried out an extensive simulation study and found that both jackknife estimators reduce the bias substantially. Although $\tilde{\rho}_{6,J}$ seems to be the best estimator in terms of bias, it has a larger mean squared error. Overall, $\tilde{\rho}_6$ seems to be the best estimator, as it possesses a small bias as well as a smaller mean squared error than that of $\tilde{\rho}_{6,J}$. For the bivariate as well as multivariate forms of the Moran–Downton exponential distribution, Balakrishnan and Ng (2001) studied the properties of estimators proposed by Al-Saadi and Young (1980) and Balakrishnan and Ng (2001a). They also used these estimators to propose pooled estimators in the multi-dimentional case and compared their performance with maximum likelihood estimators by means of Monte Carlo simulations.

Conditional Properties

The regression $E(Y\backslash X=x)$ and the conditional variance are both linear in x.

Moment Generating Function

The joint moment generating function is

$$M(s,t) = [(1-s)(1-t) - \rho st]^{-1}$$

Regression

The regression is linear and is given by

$$E(Y\backslash X = x) = 1 + \rho(x - 1)$$

Cowan's Bivariate Exponential Distribution

Formula of the Cumulative Distribution Function

The joint survival function is given by

$$\bar{H}(x, y) = exp\left[-\frac{1}{2}(x + y + \sqrt{(x + y)^2 - 4\eta xy - 4xy})\right], \qquad x, y \geq 0$$

For $0 \leq \eta \leq 1$ obviously, scale parameters can be introduced into this model if desired.

Formula of the Joint Density

The joint density function is

$$h(x, y) = \frac{1 - \eta}{2S^3}\{4\eta xy + S[S(x + y) + x^2 + y^2 + 2\eta xy]\} \exp\left[-\frac{x + y + S}{2}\right],$$
$$where S^2 = (x + y)^2 - 4\eta xy$$

Univariate Properties

Both the marginal distributions are exponential.

Correlation Coefficients

Pearson's product-moment correlation coefficient is

$$-1 + \frac{2}{\eta}\left[1 + \frac{1 - \eta}{\eta}\log(1 - \eta)\right]$$

Spearman's correlation is

$$\frac{3}{8 + \eta}\left[4 - \eta - \frac{8(1 - \eta)}{\xi} log \frac{(\eta - \xi)(3\eta + \xi)}{(\eta + \xi)(3 - \eta - \xi)}\right],$$

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Where $\xi = \sqrt{\eta(8+\eta)}$

Conditional Properties

The conditional mean and standard deviation of Y, given X=x, are not of simple form, but graphs of these functions have been given by Cowan (1987). A graph of E(Y\X=x) when the marginal have been transformed to uniforms has also been presented by Cowan.

Arnold and Strauss' Bivariate Exponential Distribution

The joint distribution was derived by Arnold and Strauss (1988).

Formula of the Joint Density

The joint density function is

$$h(x,y) = C(\beta_3)\beta_1\beta_2 e^{-\beta_1 x-\beta_2 y-\beta_1\beta_{2}xy}, \qquad x,y>0, \beta_i>0 (i=1,2), \beta_3 \geq 0,$$

where $C(\beta_3) = \int_0^\infty \frac{e^{-u}}{1+\beta_3 u} du$. Alternatively, the density may be expressed as

$$h(x,y) = K\ exp\{mxy - ax - by\},$$

where, for convergence, we must have $a,b>0$ and $m \leq 0$, and K is a normalizing constant.

Formula of the Cumulative Distribution Function

The survival function is

$$\overline{H}(x,y) = \frac{C(\beta_3)e^{-\beta_1 x-\beta_2 y-\beta_1\beta_2 xy}}{(1+\beta_1\beta_3 x)(1+\beta_2\beta_3 y)C\left(\frac{\beta_3}{(1+\beta_1\beta_3 x)(1+\beta_2\beta_3 y)}\right)}$$

Univariate Properties

Both marginal are not exponentials

Correlation Coefficient

In this case, we have $\rho \leq 0$,i.e., X and Y are negatively correlated.

Hayakawa's Bivariate Exponential

Using a finite population of exchangeable two-component systems based on the indifference principle, Hayakawa (1994) proposed a class of bivariate exponential distributions that includes the Freund, Marshall and Olkin, and Block and Basu models as special cases. For an infinite population, Hayakawa's bivariate distributions can be written as

$$\bar{H}(x,y) = \int \bar{H}\,(x, y\backslash\emptyset) dG(\emptyset),$$

Where $\bar{H}(x, y\backslash\emptyset)$ can be decomposed into an absolutely continuous part H_a and a singular part H_s, and G is the distribution function of the parameter$\emptyset$. This class of distributions includes mixtures of Freund's, Marshall and Olkin's, and Friday and Patil's distributions.

Singpurwalla and Youngren's Bivariate Exponential Distribution

Singpurwalla and Youngren (1993) introduced the following form of bivariate exponential distribution.

Formula of the Cumulative Distribution Function

The joint survival function is given by

$$\bar{H}(x,y) = \sqrt{\frac{1 - m\min(x,y) + m\max(x,y)}{1 + m(x+y)} exp\{-m\max(x,y)\}}$$

For $x, y \geq 0$, where m is a common parameter.

Formula of the Joint Density

The joint density function is

$$h(x,y) = m^2 e^{-mx} \frac{(1+mx)\{(1-mx)^2 - m^2y^2\} + \{1 + m(x-y)\}^2 - my(1+mx)}{\{1+m(x-y)\}^{3/2}\{1+m(x+y)\}^{5/2}}$$

on the sets of points $x > y$; on the set of points $y > x$, x is replaced by y and vice versa in the expression above. The joint density is undefined on the line $x = y$, which is similar to Marshall and Olkin's bivariate exponential distribution.

Univariate Properties

Both marginal distributions are exponential.

Multivariate Exponential Distributions

In this section, various forms of multivariate exponential distributions and their generalizations will be considered. These are natural extensions of the corresponding bivariate forms discussed above.

First of all, by considering n independent and identically distributed k-variate exponential random vectors with independent $Exp(\mu, \theta_i)(i = 1, \ldots, k)$ components, Bordes, Nikulin and Voinov (1997) have derived an UMVUE of the joint density function from the UMVUE of the joint distribution function. They have also illustrated the usefulness of the UMVUE of the joint density function in developing a chi-square goodness of fit for this model.

Freund's Multivariate Exponential

Weinman (1996) extended Freund's BED to the multivariate setting in the following way. Suppose a system has K identical components with times to failure $X_1, \ldots, X_k$, they all have the exponential density function

$$p_x(x) = \frac{1}{\theta_0} e^{-x/\theta_0}, \qquad x \geq 0, \theta_0 > 0.$$

It is further supposed that if ℓ components have failed (and not been replaced), the conditional joint distribution of the lifetimes of the remaining $k - \ell$components is that of independent random variables, each having the density function

$$p_x(x) = \frac{1}{\theta_\ell} e^{-x/\theta_\ell}, \qquad x \geq 0, \theta_\ell > 0.$$

In this case, clearly, $0 \leq X_1 \leq X_2 \leq \cdots \leq X_{X_k}$ then, Weinman has shown that the joint density of $X_1, \ldots, X_k$ is then

$$p_{x_1,\ldots,x_2}(x_1,\ldots,x_k) = \prod_{i=0}^{k} \frac{1}{\theta_i} e^{-(k-i)(x_{i+1}-x_i)/\theta_i}, \qquad x_0 = 0, 0 \le x_1 \le x_2 \le \cdots \le x_k$$

It is of interest to mention that the joint density function of progressively Type II right censored order statistics from an exponential distribution is a member of the family, the joint moment-generating function is

$$E[e^{t_1X_1+\cdots+t_kX_k}] = \frac{1}{k!}\sum_{p}^{*}\prod_{i=0}^{k-1}\left\{1-\frac{\theta_i}{k-i}\sum_{j=i+1}^{k} t_{p(j)}\right\}^{-1}$$

Where $\{t_{p(1)},\ldots,t_{p(k)}\}$is one of the k! Possible permutations of $t_1,\ldots,t_k$, and $\sum_p^*$ denotes the summation over all such permutations. The distribution is symmetrical in $X_1,\ldots,X_k$. for each $i(=1,2,\ldots,k)$, we have

$$E[X_i] = \frac{1}{k}\sum_{i=0}^{k-1}\theta_i,$$

$$var(X_i) = \frac{1}{k^2}[\sum_{i=0}^{k-1}\frac{k+1}{k-i}\theta_i^2 + 2\sum_{i<j}\sum i(k-i)\theta_i\theta_j],$$

and

$$cov(X_i,X_j) = \frac{1}{k^2(k-1)}\left[\sum_{i=0}^{k-1}\left(k-\frac{k+i}{k-i}\right)\theta_i^2 + 2\sum_{i<j}\sum i(k-i)\theta_i\theta_j\right]$$

The joint moment generating function of the ordered variable $0 \le X_1 \le X_2 \le \cdots \le X_{X_k}$ has the relatively simple form

$$\prod_{i=0}^{k-1}\left\{1-\frac{\theta_i}{k-i}\sum_{j=i=1}^{k} t_j\right\}^{-1}$$

Marshall and Olkin's Multivariate Exponential

Marshall and Olkin (1967) have generalized their MOBED, denoted by MOED, in the following manner. In a system of k components, the distribution of times between "fatal shocks to the combination $\{a_1,\ldots,a_\ell\}$ of components is supposed to have an exponential distribution with

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mean $1/\lambda_{a_1,\dots,a_\ell}$. The 2^{k-1}-1 different distributions of this kind are supposed to be a mutually independent set.

The resulting joint distribution of lifetimes $X_1, \dots, X_k$ of the components is

$$\bar{F}_{X_1,\dots,X_k}(x_1,\dots,x_k) = exp\left\{-\sum_{i=1}^{k}\lambda_i x_i - \sum_{i_1<i_2}\sum \lambda_{i_1,i_2}\max(x_{i_1},x_{i_2}) - \sum_{i_1<i_2<i_3}\sum\sum \lambda_{i_1,i_2,i_3}\max(x_{i_1},x_{i_2},x_{i_3}) - \cdots - \lambda_{12\dots k}\max(x_1,\dots,x_k)\right\}$$

This is also a mixed distribution, as in the bivariate case.

Arnold (1968) pointed out that estimation of the parameters $\lambda's$ by standard maximum likelihood or moment methods is not simple. He suggested the following method of estimation which exploits the singular nature of the distribution. Let

$$Z_{a_1,\dots,a_\ell} = \begin{cases} 1 & if\ X_{a_1} = \cdots = X_{a_\ell} < X_i \ \ for\ all\ i \neq a_1,\dots,a_\ell \\ 0 & otherwise. \end{cases}$$

Given n independent observations $X_j = \left(X_{1j},\dots,X_{kj}\right)^T (j = 1,\dots,n)$, each having the joint MOMED from the result of joint distribution, the estimator of $\lambda_{a_1,\dots,a_\ell}$ is, in an obvious notation,

$$\frac{\frac{1}{n}\sum_{j=1}^{n} Z_{a_1,\dots,a_\ell}(j)}{\frac{1}{n-1}\sum_{j=1}^{n}\min(X_{1j},\dots,X_{kj})}$$

The numerator and the denominator from above are mutually independent. The estimator is unbiased and has variance

$$\frac{1}{n(n-1)}\lambda_{a_1,\dots,a_\ell}\{(n-1)\lambda + \lambda_{a_1,\dots,a_\ell}\},$$

Where λ is the sum of $\lambda_{a_1,\dots,a_\ell}$'s overall possible sets $\{a_1,\dots,a_\ell\}$. However, if the sample size is not large, many of the estimators will be 0. In fact, for each X_j, only Z (at most) will not be 0, so there must be at least (2^k-1-n) estimators with 0 values.

The (k-1) dimensional marginal distributions are MOBED. Moreover, the functional equation

$$\bar{F}(x_1 + t,\dots,x_k + t) = \bar{F}(x_1,\dots,x_k)\bar{F}(t,\dots,t)$$

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Is satisfied, and the only distributions with exponential marginal distributions that satisfy above equation are the MOMEDs in resulting joint distribution.

A simplified version of MOMED is given by the survival function

$$\bar{F}_{X_1,\dots,X_k}(x_1,\dots,x_k) = exp\left\{-\sum_{i=1}^{k}\lambda_i x_i - \lambda_{k+1}\max(x_1,\dots,x_k)\right\}, \quad x_i\lambda_i > 0, \lambda_{k+1} \geq 0, \sum_{i=1}^{k+1}\lambda_i = \lambda$$

Symmetry corresponds to $\lambda_1 = \cdots = \lambda_k$that is, $\gamma_i = \lambda_i - \lambda_k = 0 \ (i = 1,\dots,k-1)$ while mutual independence corresponds to $\lambda_{k+1} = 0$.

Block and Basu's Multivariate Exponential

This model is an extension of the ACBED of Block and Basu (1974), to the multivariate case and constitutes the absolutely continuous part of the MOMED, if $X=(X_1,\dots,X_2)^T$ represents the joint lifetime of k components, the corresponding $(k+1)$ parameter density function is

$$P_X(x) = \frac{\lambda_{i_1} + \lambda_{k+1}}{\alpha}\prod_{r=2}^{k}\lambda_i\bar{F}_M(x), \quad x_{i_1} > \cdots > x_{i_k}, \quad i_1 \neq i_2 \neq \cdots \neq i_k = 1,2,\dots,k,$$

Where

$$\bar{F}_M(x) = exp\left\{-\sum_{r=1}^{k}\lambda_{i_r}x_{i_r} - \lambda_{k+1}x_{(k)}\right\}, \quad \alpha = \sum\dots\sum\frac{\prod_{r=2}^{k}\lambda_{i_r}}{\prod_{r=2}^{k}(\sum_{j=1}^{r}\lambda_{i_j} + \lambda_{k+1}}$$

And $x_{(k)}$ is max $(x_1,\dots,x_k)$.

The failure times $X_1,\dots,X_k$ are independent if $\lambda_{k=1} = 0$ the condition $\lambda_1 = \cdots = \lambda_k$ implies symmetry and it is equivalent to identical marginal of all the k components. The model in parameter density function satisfies the lack of memory property, but the marginal are weighted combinations of exponentials.

Let $X_1,\dots,X_n$ be a random sample from parameter density function. Let n_{i1} denote the number of observations with $X_{i_1} > \max(x_{i_2},\dots,x_{i_k})$ the expected value of n_{i1} is

$$E[n_{i_1}] = \frac{n}{\alpha}\sum_{i_2\neq\cdots\neq i_k=1}^{k}\cdots\sum\prod_{r=2}^{k}\frac{\lambda_{i_r}}{\sum_{j=1}^{r}\lambda_{i_j} + \lambda_{k+1}}.$$

The likelihood equations are

$$\frac{\partial \log L}{\partial \lambda_{i_1}} = -n\,\alpha_{i_1} + \frac{n_{i_1}}{\lambda_{i_1} + \lambda_{k+1}} + \frac{n - n_{i_1}}{\lambda_{i_1}} - \sum_{j=1}^{n} X_{i_1 j} = 0, \quad i_1 = 1, 2, \ldots, k,$$

$$\frac{\partial \log L}{\partial \lambda_{k+1}} = -n\,\alpha_{k+1} + \sum_{i_1=1}^{k} \frac{n_{i_1}}{\lambda_{i_1} + \lambda_{k+1}} - \sum_{j=1}^{n} X_{(k)j} = 0,$$

Where $\alpha_{i_1} = \frac{\partial log\alpha}{\partial \lambda_{k+1}}$, $i_1 = 1, \ldots, k+1$

Fisher information matrix

$$nI(\lambda) = \left((nI_{ij})\right) = \left(\left(E\left[\frac{\partial^2 \log L}{\partial \lambda_i \partial \lambda_j}\right]\right)\right), \qquad i,j = 1, \ldots, k+1$$

Is positive definite in this case and $\sqrt{n}(\hat{\lambda} - \lambda)$ has asymptotic multivariate normal distribution with mean vector 0 variance-covariance matrix $I^{-1}(\lambda)$.

Olkin and Tong's Multivariate Exponential

Olkin and Tong (1994) studied an important subclass of MOMEDs. Let $U_1, \ldots, U_k$, $V_1, \ldots, V_k$ and W be independent exponential random variables with $E[U_i] = {}^1/_{\lambda_1}$, $E[V_i] = {}^1/_{\lambda_2}$ $(i = 1, \ldots, k)$ and $E[W] = {}^1/_{\lambda_0}$. Let $K = (K_1, \ldots, K_k)^T$ be a vector of non-negative integers with

$$\sum_{s=1}^{k} K_s = k, \qquad K_1 \geq \cdots \geq K_r \geq 1, \;\; K_{r+1} = \cdots = k_k = 0$$

For some $r \leq k$. For given K, let $X(K) = (X_1, \ldots, X_k)^T$ be a k-dimensional multivariate exponential random variable defined by

$$\begin{cases} \min(U_j, V_1, W), & j = 1, \ldots, K_1 \\ \min(U_j, V_2, W), & j = K_{1+}1, \ldots, K_1 + K_2 \\ & \vdots \\ \min(U_j, V_r, W), & j = K_{1+}, \ldots, K_{r-1} + 1, \ldots, k \end{cases}$$

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Note that the distribution of $(X_1, \dots, X_k)^T$ belongs to a subclass of the MOMED family. The latter, requires 2^k-1 independent variables to generate a k-variance exponential distribution. The univariate marginal distributions of X_j's are exponential with mean $1/(\lambda_0 + \lambda_1 + \lambda_2)$.

The joint distribution of X_i's is exchangeable when $K=(k,0,\dots,0)^T$ and also when $K = (1, \dots, 1)^T$. The components $X_j = \min(U_j, V_r, W)$, $j = 1, \dots, n$, of $X(1, \dots, 1)$.

For a fixed but arbitrary $k, \lambda = (\lambda_0, \lambda_1, \lambda_2)^T$ and t, let K and K' be two vectors satisfying above equation. $K > K'$ where > denotes majorization order, then Olkin and Tong (1994) have established that

$$\bar{F}_{K,\lambda}(t, \dots, t) \geq \bar{F}_{\acute{K},\lambda}(t, \dots, t)$$

Moran and Downton's Multivariate Exponential

Al-Saddi and Young (1982) generalized moran and Downton's bivariate exponential distribution, to the equicorrelated multivariate case as follows. Let $X_i = \sum_{j=1}^{M} Y_{ij}$ where Y_{ij}'s are independent and identically distributed random variables with density function

$$p_{Y_i}(y) = \frac{\theta_i}{1-\rho} e^{-\theta_i y/(1-\rho)}, \qquad y > 0, i = 1, \dots, k$$

Let M have a geometric distribution with probability mass function

$$P_r[M = m] = (1-\rho)\rho^{m-1}, \qquad 0 \leq \rho < 1, \qquad m = 1,2, \dots$$

Then, conditional on $M=m$, the distribution of X_i is gamma with probability density function

$$f_i(x) = \left(\frac{\theta_i}{1-\rho}\right)^m \frac{x^{m-1}}{(m-1)!} e^{-\theta_i x/(1-\rho)}$$

and the joint unconditional density function of $X=(X_1, \dots, X_k)^T$ is

$$p_X(x) = \sum_{m=1}^{\infty} P_r[M = m] \prod_{i=1}^{k} f_i(x_i)$$

$$= \frac{\theta_1 \dots \theta_k}{(1-\rho)^{k-1}} exp\left\{-\frac{1}{1-\rho}\sum_{i=1}^{k} \theta_i x_i\right\} S_k\left(\frac{\rho\theta_1 x_1 \theta_2 x_2 \dots \theta_k x_k}{(1-\rho)^k}\right), \qquad x_i > 0, \ i = 1, \dots, k,$$

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where $S_k(z) = \sum_{i=0}^{\infty} z^i/(i!)^k$.

The marginal distribution of X_i is exponential with parameter $\theta_i (i = 1, \ldots, k)$. Nothing that $I_0(z)$ the modified Bessel function of the first kind of order zero- is $I_0(z) = S_2(z^2/4)$ we observe that the unconditional density function reduces readily to the bivariate Moran and Downton's density, the mixed moment of order $(r_1, \ldots, r_k)$ is

$$E\left[X_1^{r1} \ldots X_k^{rk}\right] = \sum_{j1=0}^{r1} \ldots \sum_{jk=0}^{rk} \frac{(r-j)!\,\rho^{r-j}(1-\rho)^j}{\theta_1^{r1}\theta_2^{r2} \ldots \theta_k^{rk}} \prod_{i=1}^{k} \left[\binom{r_i}{j_i} \frac{\{r_i + \sum_{\ell-1}^{i-1}(r_\ell - j_\ell)\}!}{\{\sum_{\ell=1}^{i}(r_\ell - j_\ell)\}!}\right]$$

Where $r = \sum_{\ell=1}^{k} r_\ell$, $j = \sum_{\ell=1}^{k} j_\ell$, and $r_0 = j_0 = 0$. In particular, setting $r_s = r_t = 1$ and $r_i = 0$, for $i \neq s,t$ we obtain

$$E[X_s X_t] = \frac{1+\rho}{\theta_s \theta_t}, \quad s = 1, \ldots, k-1; \quad t = s+1, \ldots, k$$

Which shows that each pair of random variables has correlation coefficient equal t ρ.

Raftery's Multivariate Exponential

Raftery (1984) and O'Cinneide and Raftery (1989) studied a multivariate exponential distribution which is defined as follows. Suppose that $Y_1, \ldots, Y_k$ and $Z_1, \ldots, Z_\ell$are independent exponential (λ) random variables and $(J_1, \ldots, J_k)$ is a random vector taking on values in $\{0,1, \ldots, \ell\}^k$ with marginal

$$P_r[J_i = 0] = 1 - \pi_i \text{ and } P_r[J_i = j] = \pi_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, \ell,$$

where $\pi_i = \sum_{j=1}^{\ell} \pi_{ij}$. let $Z_0 \equiv 0$. then, the model for $X_1, \ldots, X_k$ is

$$X_i = (1 - \pi_i) Y_i + Z_{Ji}, \qquad i = 1, \ldots, k$$

The main properties of this model are similar to those of the multivariate normal distribution in the sense that univariate marginal are exponential while bivariate marginal belong to Raftery's bivariate exponential distribution, given by

$$X_i = (1 - \pi_i) Y_i + I_i Z, \quad i = 1,2,$$

a linear combination of the underlying independent random variables. Here, Y_1,Y_2, and Z are independent exponential (λ) random variables, and I_i's (i=1,2) are binary 0-1 random variables with

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$$P_r[I_i = 1] = \pi_i \quad i = 1,2 \quad \text{and} \quad P_r[I_1 = i, I_2 = j] = p_{ij} \quad i,j = 0,1$$

When $\ell = 1, p_{11} = P_r[J_i = J_i = 1]$ and moreover

$$\rho_{ij} = corr(X_i, X_j) = \alpha_{ij} + \beta_{ij} + \pi_i + \pi_j - \pi_i\pi_j - 1$$

with $\alpha_{ij} = P_r[J_i = J_j = 0]\ and\ \beta_{ij} = P_r[J_i = J_j \neq 0]$ so that the correlation structure is independent of the marginal distributions. Unfortunately, the dependence structure involves $(\ell + 1)^k - 1$ parameters. Raftery (1984) has therefore recommended to constrain the bivariate marginal distributions to be exchangeable.

In the bivariate case, Nagaraja and Baggs (1996) have discussed the joint and marginal distributions of order statistics $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$ as well as some reliability properties of these order statistics.

Krishnamoorthy and Parthasarathy's Multivariate Exponential

A futher example of a multivariate exponential distribution can be obtained by taking v=2 in the multivariate gamma distribution of Krishnamoorthy and Parthasarathy (1951); the joint characteristic function is

$$E\left[e^{i(t_1X_1+\cdots+t_kX_k)}\right] = |I_k - 2iRD_t|^{-1}$$

Where R is a correlation matrix, I_k is an identity matrix of order k, and D_t=$Diag(t_1,\ldots,t_k)$. since $|I_k - 2iRD_t|$ is a polynomial in $(1 - 2it_1), \ldots, (1 - 2it_k)$, the joint distribution of $(X_1,\ldots,X_k)^T$ can be expressed formally as a mixture of a finite number of x^2-distributions.

By considering two independent copies of Krishnamoorthy and Parthasarathy's multivariate gamma variables of index $\frac{1}{2}$, and adding them, one could obtain a multivariate exponential distribution. Kent (1983) has shown the equivalence of the distribution so obtained and the distribution derived from considering the sojourn time vector of a birth-death process up to a first passage time. Recall that in the univariate case, we have two derivations of exponential distributions-one based on the lack of memory property which is equivalent to the waiting time spent in a given state of continuous time Markov process before jumping into a new state, and the other, based on the normal distribution, as the distribution of $X_1^2 + X_2^2$when X_1 and X_2 are independent normal random variables with zero mean and same X_2 are independent normal random variables with zero mean and same variance.

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