# Domination and Global Domination Of Some Operations On Product Fuzzy Graphs

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**Abstract**

In this paper, Some operation in product fuzzy graphs are introduced and investigated. The domination $\gamma(G)$ and global domination $\gamma_g(G)$ number of such operations are introduced and some bounds of $\gamma(G)$ and $\gamma_g(G)$ were given with some suitable examples.

**Keyword:** Union on product fuzzy graph, Intersection on product fuzzy graph, join on product fuzzy graph and domination number on Some operation in product fuzzy graphs.

## 1 Introduction

The "Fuzzy sets" firstly was introduced and investigated In(1965)[13] by L.A. Zadeh which described fuzzy set theory and consequently, fuzzy logic.
This theory essentially proposes graded membership for every element in a subset of a universal set by assigning a particular value for every element in the closed interval [0,1] this value is called the membership degree of such element.
Zadeh's ideas have been applied to a wide range of scientific areas such as computer science, artificial intelligence, decision analysis, information science, system science, control engineering, expert systems, pattern recognition, management science, operation research.
So also many areas of mathematics have been touched by fuzzy set theory.
The ideas of fuzzy set theory have been introduced into topology, abstract algebra, geometry, graph theory and analysis.

The fuzzy relation was also investigated by L.A. zadeh in (1987)[14].
The notion of fuzzy graph and several fuzzy analogues of graph theoretic concepts such as path, cycles and connectedness were investigated by Rosenfeld(1975)[8].
The mathematical formal definition of domination in graph was investigated by Ore(1962)[7].
Mordeson and Nair(1996)[4] introduced the concept of cycles and cocycles of fuzzy graphs. Fuzzy cycle and Fuzzy trees was investigated by Mordeson and Y. Y. Yao(2002)[6].
The concepts of domination in fuzzy graphs was investigated by A. Somasundaram, S. Somasundaram [11].
The concept of product of fuzz graphs given by V. Ramaswamy (2009)[9].
The first definition of global dominating sets in graphs was introduced by Sampathkumar in (1989)[10].
The concepts of global domination number, domatic number and global domatic number in fuzzy graph was introduced by Mahioub shubatah(2009)[1].
Mahioub shubatah (2012)[2] introduced the concept of domination in product of fuzzy graph.
An operations on fuzzy graph were been introduced and studied by Mordeson, J.N and Peng in 1994[5] and also by Venugopalam, Naga Maruthi Kumari in 2017[12].
Mahioub M.Q shubatah and Haifa A. A (2020)[3] introduced the concept of global domination in product fuzzy graph..
In this research we introduce and study some operations in product fuzzy graph such as: Union on product fuzzy graph, Intersection on product fuzzy graph and join on product fuzzy graph.
We also discuss some results and bounds of $\gamma(G)$ and $\gamma_g(G)$ on some operations of product fuzzy graph.

## 2 Definitions

In this section, we review briefly some definitions in Graphs, fuzzy graphs, product fuzzy graphs, and domination number in a product fuzzy graph.
A crisp graph G is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges.
The vertex sets and the edges set of G are denoted by V(G) and E(G) respectively.
A fuzzy graph G = $(\mu,\rho)$ is a set $V$ with two function $\mu$ :V $\rightarrow$ [0,1] and $\rho:$ E $\rightarrow$ [0,1] such that $\rho(\{u,v\}) \leq \mu(u) \wedge \mu(v)$ for all $u,\ v \in V$. We write $\rho(\{u,\ v\})$ for $\rho(u,\ v)$.
The order $p$ and size $q$ of a fuzzy graph $G = (\mu,\ \rho)$ are defined to be $p = \sum_{u\in V} \mu(u)$ and $q = \sum_{(u,\ v)\in E} \rho(u,\ v)$.

Let $G = (\mu,\ \rho)$ be any fuzzy graph, then the complement of $G$ denoted by $\overline{G}$ and is defined as $\overline{G} = (\overline{\mu},\overline{\rho})$, where $\overline{\mu}(v) = \mu(v)\ \forall\ v\ \in V(G)$ and $\overline{\rho}(u,v) =$

$1-\rho(u,v),\ \forall\ u,\ v \in V(G)$.
Let $G=(\mu,\ \rho)$ be any fuzzy graph on $V$ and $u,\ v \in V(G)$, then we say that $u$ and $v$ dominates each other if $\rho(u,\ v)=\mu(u)\wedge\mu(v)$.
A vertex subset $D$ of $V$ is called a dominating set of a fuzzy graph $G$ if for every $v \in V-D$ there exists $u \in D$ such that $u$ dominates $v$.
A dominating set $D$ of a fuzzy graph $G$ is called minimal dominating set if $D-\{v\}$ is not dominating set of G for all $v \in D$.
The minimum fuzzy cardinality taken over all minimal dominating set in fuzzy graph $G$ is called the domination number of $G$ and denoted by $\gamma(G)$.
A vertex subset $D$ of $V$ in a fuzzy graph $G$ is said to be global dominating set of $G$ if $D$ is a dominating set in bath $G$ and $\overline{G}$.
Let $G$ be a graph whose vertex set is $V, \mu$ be a fuzzy subset of $V$ and $\rho$ be a fuzzy subset of V× V, we call $(\mu,\rho)$ a product partial fuzzy subgraph of G(in short, a product fuzzy graph) $if \rho(u,v) \leq \mu$(u)× $\mu$(v) for all $u,v \in$ V.
A product fuzzy graph $G=(\mu,\rho)$ is called complete product fuzzy graph if $\rho(u,v)=\mu(u)\times\mu(v)$ for all $u,v \in V$.
A product fuzzy graph $G$ is said to be bipartite product fuzzy graph if its vertex set $V$ can be partitioned in to two nonempty subsets $V_1$ and $V_2$ such that $\rho(u,v)=0$ if $u,v \in V_1$ or $u,v \in V_2$.
We say that a bipartite product fuzzy graph $G$ is a complete bipartite product fuzzy graph if $\rho(\{u,v\})=\mu$ (u)×$\mu(v)$ for all $u \in V_1$, $v \in V_2$.
The complement of a product fuzzy graph $G=(V,\mu,\rho)$ is denoted by $\overline{G}=(V,\overline{\mu},\overline{\rho})$ where $\mu=\overline{\mu}$ and $\overline{\rho}\ (u,v)=\mu(u)\times\mu(v)-\rho(u,v)$.
Let $G=(V,\mu,\rho)$ be a product fuzzy graph and $u,v \in V(G)$. Then we say $u$ dominates $v$ if $\rho(u,v)=\mu(u)\times\mu(v)$.
Let $G=(V,\mu,\rho)$ be a product fuzzy graph then a vertex subset D of V(G) is said to be dominating set of G if for every vertex $v \in (V-D)$ there exists a vertex $u \in D$ such that $\rho(u,v)=\mu(u)\times\mu(v)$.
The dominating set D of a product fuzzy graph is called minimal product dominating set if $D-\{v\}$ is not dominating set of $G$, for all vertex $v$ in $D$.
The minimum fuzzy cardinality taken over all minimal dominating sets of a product fuzzy graph $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$.
Two vertices $u$ and $v$ in a product fuzzy graph $G=(\mu,\rho)$ are said to be an independent if $\rho(u,v)<\mu(u)\wedge\mu(v)$ or $\rho(u,v)=0$.
A vertex subset $D$ of $V$ in a product fuzzy graph $G=(\mu,\rho)$ is called an independent vertex set if all vertices in $D$ are independent.
An independent vertex set $D$ in a product fuzzy graph $G$ is said to be maximal independent if $D \cup \{u\}$ for all $u \in (V-D)$ is not independent.
The maximum fuzzy cardinality taken over all an independent sets of a product fuzzy graph is called an independence number and is denoted by $\beta_0(G)$.
If $e=(u,v)$ is an edge in a product fuzzy graph $G$ then we say that $u$ and $v$ cover the edge $e$ and $e$ covers $u$ and $v$.

A vertex subset $D$ of $V$ in a product fuzzy graph $G$ is called vertex cover set of a product fuzzy graph $G$ if for all edge $e$ in $G$ their is a vertex $v$ in $D$ such that $v$ covers $e$.
The minimum fuzzy cardinality taken over all vertex cover sets of a product fuzzy graph $G$ is called the vertex covering number of $G$ and is denoted by $\alpha_0(G)$.
A dominating set $D$ of a product fuzzy graph $G = (V, \mu, \rho)$ is called connected dominating set of $G$ if the fuzzy subgraph $< D >$ induced by $D$ is connected.
The connected domination number of a product fuzzy graph G is the minimum cardinality taken over all connected dominating sets in G and is denoted by $\gamma_c(G)$.
A dominating set $D$ of a product fuzzy graph $G$ is called an independent dominating set if $D$ is an independent.
The independence domination number of a product fuzzy graph $G$ is the minimum fuzzy cardinality taken over all an independent dominating sets in $G$ and is denoted by $\gamma_i(G)$.
Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where $E'$ is the set of all arcs joining the nodes of $V_1$ and $V_2$ where we a assume that $V_1 \cup V_2 = \phi$ then the join of two fuzzy graphs $G_1$ and $G_2$ is a fuzzy graph $G = G_1 + G_2 : (\mu_1 + \mu_2, \rho_1 + \rho_2)$ defined as follows:

$$\mu_1 + \mu_2 = \begin{cases} (\mu_1 \cup \mu_2) & if\ u \in V_1 \cup V_2 \\ \mu_1(u);\ \ u \in V_1 - V_2 \\ \mu_2(u);\ u \in V_2 - V_1 \end{cases}$$

and

$$\rho_1 + \rho_2 = \begin{cases} (\rho_1 \cup \rho_2) & if\ (u,\ v) \in E_1 \cup E_2 \\ \rho_1(u,\ v);\ \ (uv) \in E_1 - E_2 \\ \rho_2(u,\ v);\ \ (u,\ v) \in E_2 - E_1 \\ \mu_1 \times \mu_2 & if (u,\ v) \in E' \end{cases}$$

Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the intersection $G^* = G_1^* \cap G_2^* = (V_1 \cap V_2, E_1 \cap E_2)$ of graphs. We a assume that $V_1 \cap V_2 \neq \phi$ then the intersection of two fuzzy graphs $G_1$ and $G_2$ is a Product fuzzy graph $G = G_1 \cap G_2 : (\mu_1 \cap \mu_2\ ,\ \rho_1 \cap \rho_2)$ defined as follows:

$$\mu_1 \cap \mu_2 = \{\ min(\mu_1, \mu_2) \quad if\ u \in V_1 \cap V_2.$$

and

$$\rho_1 \cap \rho_2 = \{\ min(\rho_1, \rho_2) \quad if\ (uv) \in E_1 \cap E_2$$

Let $G_1 = (\mu_1, \rho_1)$ and $G_2 = (\mu_2, \rho_2)$ be two fuzzy graphs consider the union $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ of graphs, where $V_1 \cup V_2 = \phi$ then the union of two fuzzy graphs $G_1$ and $G_2$ is a fuzzy graph $G = G_1 \cup G_2 : (\mu_1 \cup \mu_2\ ,\ \rho_1 \cup \rho_2)$ defined as follows:

$$\mu_1 \cup \mu_2 = \begin{cases} max(\mu_1, \mu_2) & if\ u \in V_1 \cup V_2 \\ \mu_1(u);\ u \in V_1 - V_2 \\ \mu_2(u);\ u \in V_2 - V_1 \end{cases}$$

and

$$\rho_1 \cup \rho_2 = \begin{cases} max(\rho_1, \rho_2)(u,\ v) & if\ (u,\ v) \in E_1 \cup E_2 \\ \rho_1(u,\ v);\ (u,\ v) \in E_1 - E_2 \\ \rho_2(u,\ v);\ (u,\ v) \in E_2 - E_1 \end{cases}$$

## 3 Some Operations on product fuzzy graphs

In this section we introduce and study Some Operations on a product fuzzy graphs such as the intersection, the join and the union

**Definition 1.** *Let* $G_1 = (\mu_1,\ \rho_1)$ *and* $G_2 = (\mu_2,\ \rho_2)$ *be two Product fuzzy graph consider the intersection* $G^* = G_1^* \cap G_2^* = (V_1 \cap V_2,\ E_1 \cap E_2)$ *of graphs. We a assume that* $V_1 \cap V_2 \neq \phi$ *then the intersection of two Product fuzzy graph* $G_1$ *and* $G_2$ *is a Product fuzzy graph* $G = G_1 \cap G_2 : (\mu_1 \cap \mu_2,\ \ \rho_1 \cap \rho_2)$ *defined as follows:*

$$\mu_1 \cap \mu_2 = \{\ min(\mu_1,\ \mu_2) \quad if\ \ u \in V_1 \cap V_2.$$

*and*

$$\rho_1 \cap \rho_2 = \{\ min(\rho_1,\ \rho_2) \quad if\ \ (uv)\ \ \in E_1 \cap E_2$$

**Example 2.** *Consider the two product fuzzy graphs* $G_1$ *and* $G_2$ *given in Fig. 3.1(a), 3.1(b) respectively, where* $G_1 = (V_1,\ \mu_1,\ \rho_1)$ *such that* $V_1 = \{u_6,\ u_1,\ u_2,\ u_3,\ u_5\},\ \ \mu_1(u_6) = 0.4,\ \ \mu_1(u_1) = 0.1,\ \ \mu_1(u_2) = 0.3,\ \ \mu_1(u_3) = 0.2,\ \ \mu_1(u_5) = 0.6$ *and* $\rho(u,\ v) = \mu_1(u) \times \mu_1(v)$ *for all* $u,\ v \in E_1$ *and* $G_2 = (V_2,\ \mu_2,\ \rho_2)$ *such that* $V_2 = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5\},\ \mu_2(u_1) = 0.2,\ \ \mu_2(u_2) = 0.1,\ \ \mu_2(u_3) = 0.5,\ \ \mu_2(u_4) = 0.2,\ \ \mu_2(u_5) = 0.6,$ *and* $\rho(u,\ v) = \mu_2(u) \times \mu_2(v)$ *for all* $u, v \in E_2$*. Then the intersection* $(G_1 \cap G_2)$ *given in Fig. (3.1(c)).*

$G_1$ :

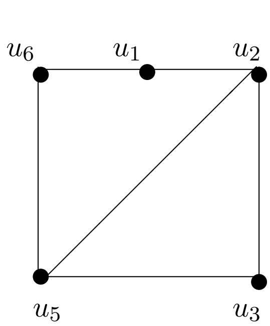



*Fig.* 3.1(*a*)

$G_2$ :

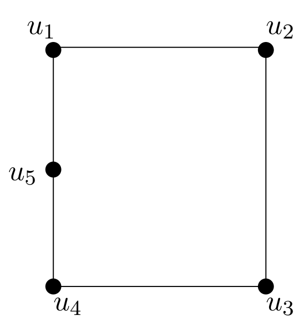
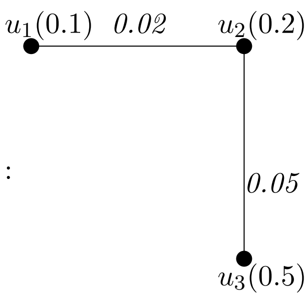



*Fig.* 3.1(*b*)

$G_1 \cap G_2$ :




$Fig.\ 3.1(c)$

In the following theorem we give $\gamma$ and $\gamma_g$ of the intersection of any disjoint product fuzzy graphs $G_1$ and $G_2$.

**Theorem 3.** *Let $G_1$ and $G_2$ be two disjoint product fuzzy graphs, then*

$$\gamma(G_1 \cap G_2) = 0$$

*Proof.* Let $D_1$ be a $\gamma - set$ of a product fuzzy graph $G_1$ and let $D_2$ be a $\gamma - set$ of a product fuzzy graph $G_2$. Since $G_1$ and $G_2$ are disjoint then $D_1 \cap D_2 = \phi$. Therefor $\gamma(G_1 \cap G_2) = |D_1 \cap D_2| = |\phi| = 0$. □

**Theorem 4.** *Let $G_1$ and $G_2$ be two disjoint product fuzzy graphs, then*

$$\gamma_g(G_1 \cap G_2) = 0$$

*Proof.* Let $D_1$ be a $\gamma_g - set$ of a product fuzzy graph $G_1$ and Let $D_2$ be a $\gamma_g - set$ of a product fuzzy graph $G_2$. Since $G_1$ and $G_2$ are disjoint then $D_1 \cap D_2 = \phi$. Therefor $\gamma_g(G_1 \cap G_2) = |D_1 \cap D_2| = |\phi| = 0$. □

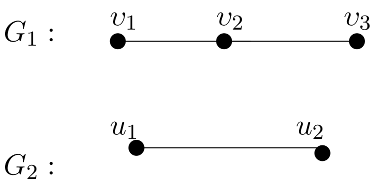
**Definition 5.** *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ be two Product fuzzy graph consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2,\ E_1 \cup E_2 \cup E')$ of graphs where $E'$ is the set of all edges joining the vertices of $V_1$ and $V_2$ where we a assume that $V_1 \cup V_2 = \phi$ then the join of two Product fuzzy graph $G_1$ and $G_2$ is a Product fuzzy graph $G = G_1 + G_2 : (\mu_1 + \mu_2,\ \rho_1 + \rho_2)$ defined as follows:*

$$\mu_1 + \mu_2 = \begin{cases} (\mu_1 \cup \mu_2) & if\ u \in V_1 \cup V_2 \\ \mu_1(u);\ u\ \in V_1 - V_2 & \\ \mu_2(u);\ u \in V_2 - V_1 & \end{cases}$$

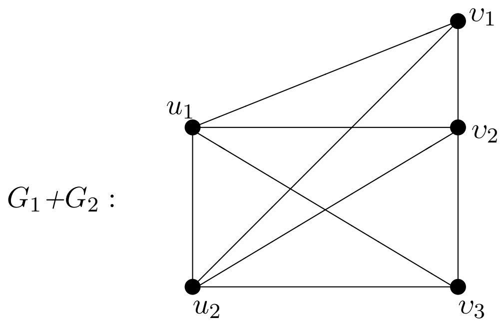
*and*

$$\rho_1 + \rho_2 = \begin{cases} (\rho_1 \cup \rho_2) & if\ (u,\ v) \in E_1 \cup E_2 \\ \rho_1(u,\ v);\ (u,\ v) \in E_1 - E2 & \\ \rho_2(u,\ v);\ (u,\ v) \in E_2 - E_1 & \\ \mu_1 \times \mu_2 & if\ (u,\ v) \in E' \end{cases}$$

**Example 6.** *Consider the two product fuzzy graphs* $G_1$ *and* $G_2$ *given in Fig. 3.2(a) where,* $G_1 = (\mu_1,\ \rho_1)$ *defined as:* $\mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ *and* $\rho(u,\ v) = \mu_1(u) \times \mu_1(v)$ *for all* $u, v \in E_1$ *and* $G_2 = (\mu_2,\ \rho_2)$ *defined as:* $\mu_2(u_1) = 0.4,\ \mu_2(u_2) = 0.3$ *and* $\rho(u,\ v) = \mu_2(u) \times \mu_2(v)$ *for all* $u, v \in E_2$*. Then the Fig.* $3.2(b)$) *gives the Product fuzzy graph* $(G_1 + G_2) = (V,\ \mu,\ \rho)$ *where,* $V = \{u_1,\ u_2,\ v_1,\ v_2,\ v_3\}$ *defined as:* $\mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \mu(v_3) = 0.3,\ \mu(u_1) = 0.4,\ \mu(u_2) = 0.3,\ \rho(u_1,\ v_3) = 0.12,\ \rho(u_1,\ u_2) = 0.12,\ \rho(u_1,\ v_2) = 0.04,\ \rho(u_1,\ v_1) = 0.08,\ \rho(u_2,\ v_1) = 0.08,\ \rho(u_2,\ v_2) = 0.03,\ \rho(u_2,\ v_3) = 0.09.$



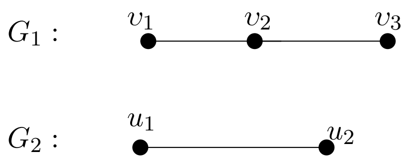

$Fig.\ 3.2(a)$



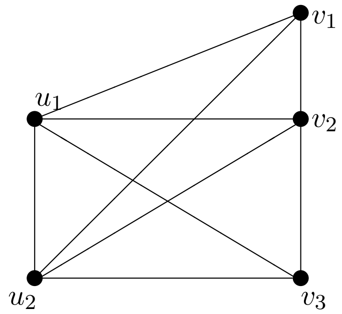

$Fig.\ 3.2(b)$

**Theorem 7.** *Let* $G_1 = (\mu_1,\ \rho_1)$ *and* $G_2 = (\mu_2,\ \rho_2)$ *be two Product fuzzy graphs then,* $\overline{G_1 + G_2} \neq \overline{G_1} + \overline{G_2}$.

**Example 8.** *Consider the product fuzzy graphs* $G_1 = (V_1,\ \mu_1,\ \rho_1)$ *where,* $V_1 = \{v_1,\ v_2,\ v_3\},\ \mu_1(v_1) = 0.2,\ \mu_1(v_2) = 0.1,\ \mu_1(v_3) = 0.3$ *and* $\rho(u,\ v) = \mu_1(u) \times \mu_1(v)$ *for all* $u, v \in E_1$, $G_2 = (V_2,\ \mu_2,\ \rho_2)$ *where,* $V_2 = \{u_1, u_2\},\ \mu_2(u_1) = 0.4,\ \mu_2(u_2) = 0.3,\ \rho(u_1, u_2) = 0.12,\ G = (G_1 + G_2) = (V,\ \mu,\ \rho)$ *where,* $V = V_1 \cup V_2 = \{u_1, u_1, v_1, v_2, v_3\},\ \mu(v_1) = 0.2,\ \mu(v_2) = 0.1,\ \mu(v_3) = 0.3,\ \mu(u_1) = 0.4,\ \mu(u_2) = 0.3,\ \rho(u_1, v_1) = 0.08,\ \rho(u_1, v_2) = 0.04,\ \rho(u_1, v_3) = 0.12,\ \rho(u_1, u_2) = 0.12,\ \rho(u_2, v_1) = 0.06,\ \rho(u_2, v_2) = 0.03,\ \rho(u_2,\ v_3) = 0.09,\ \rho(v_1, v_2) = 0.02,$ $\overline{G_1 + G_2} = (V, \overline{\mu}, \overline{\rho})$ *where,* $V = \{u_1, u_1, v_1, v_2,\ v_3\},\ \overline{\mu}(v_1) = 0.2,\ \overline{\mu}(v_2) = 0.1,\ \overline{\mu}(v_3) = 0.3,\ \overline{\mu}(u_1) = 0.4,\ \overline{\mu}(u_2) = 0.3,\ \overline{\rho}(v_1, v_3) = 0.06$ *and* $\overline{\rho}(v_1,\ v_2) = \overline{\rho}(u_1,\ v_3) = \overline{\rho}(u_2,\ v_2) = \overline{\rho}(u_2,\ v_3) = \overline{\rho}(u_2,\ v_1) = \overline{\rho}(u_1,\ v_3) = \overline{\rho}(u_1,\ v_1) = \overline{\rho}(u_1,\ v_2) = \overline{\rho}(u_1,\ u_2) = 0,\ \overline{G_1} = (V, \overline{\mu}_1, \overline{\rho}_1)$ *where,* $V = \{v_1,\ v_2,\ v_3\},\ \overline{\mu}_1(v_1) = 0.2,\ \overline{\mu}_1(v_2) = 0.1,\ \overline{\mu}_1(v_3) = 0.3,\ \rho(v_1,\ v_3) = 0.06, \overline{G_2} = (V,\ \overline{\mu}_2,\ \overline{\rho}_2)$ *where,* $V_2 = \{u_1, u_2\},\ \overline{\mu}_1(u_1) = 0.4,\ \overline{\mu}_1(u_2) = 0.3$ *and* $\overline{\rho}_2(u_1,\ u_2) =$ *0,*
*which given in Fig.* $3.3(a),\ 3.3(b),\ 3.3(c),\ 3.3(d)$*, respectably . Finally* $G = \overline{G_1} + \overline{G_2} = (V, \overline{\mu}, \overline{\rho})$ *given in Fig. 3.3(e).*

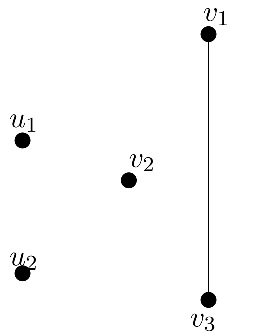



$Fig.\ 3.3(a)$



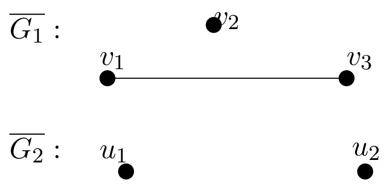

$G_1+G_2$ :

$Fig.\ 3.3(b)$

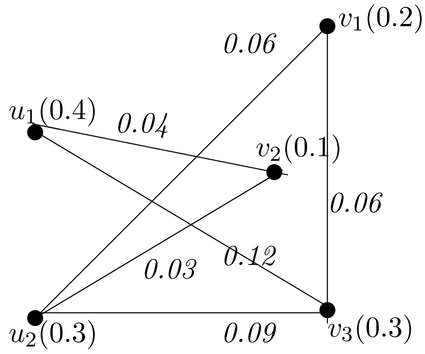



$\overline{G_1+G_2}$

$Fig.\ 3.3(c)$




$Fig.\ 3.3(d)$




$\overline{G_1}+\overline{G_2}$

$Fig.\ 3.3(e)$

In the following theorem we give $\gamma_g$ of the join of any complete product fuzzy graphs $G_1$ and $G_2$.

**Theorem 9.** *If $G = G_1 + G_2 = (\mu_1 + \mu_2,\ \rho_1 + \rho_2)$ is a complete product fuzzy graph then,*

$$\gamma_g(G_1 + G_2) = p \quad where, p = p_1 + p_2.$$

*Proof.* Let $G = G_1 + G_2$ be a complete product fuzzy graphs, then every vertex of $G$ has $(n-1)$ neighbors. Since the complement of $G$ is the null graph then $V$ is only the global dominating set of $G$ and $\overline{G}$. Hence $\gamma_g(G) = p$.

□

**Theorem 10.** *If $G = G_1 + G_2 = (\mu_1 + \mu_2,\ \rho_1 + \rho_2)$ is a complete product fuzzy graph then,*

$$\gamma_g(G_1 + G_2) = \gamma_g(\overline{G_1 + G_2})$$

*Proof.* Let $G = G_1 + G_2 = (\mu_1 + \mu_2,\ \rho_1 + \rho_2)$ be any product fuzzy graph and $D$is a minimal global dominating set then, $D$ is a dominating set of $G$ and $\overline{G}$, clearly $\gamma_g(G) = \gamma_g(\bar{G})$.

□

**Definition 11.** *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ be two Product fuzzy graphs consider the union $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2,\ E_1 \cup E_2)$ of graphs where we a assume that $V_1 \cup V_2 = \phi$ then the union of two Product fuzzy graphs $G_1$ and $G_2$ is a Product fuzzy graph $G = G_1 \cup G_2 : (\mu_1 \cup \mu_2,\ \rho_1 \cup \rho_2)$ defined as follows:*

$$\mu_1 \cup \mu_2 = \begin{cases} max\ (\mu_1,\ \mu_2) & if\ u \in V_1 \cup V_2 \\ \mu_1(u);\ u \in V_1 - V_2 \\ \mu_2(u);\ u \in V_2 - V_1 \end{cases}$$

*and*

$$\rho_1 \cup \rho_2 = \begin{cases} max\ (\rho_1,\ \rho_2) & if\ (u,\ v) \in E_1 \cup E_2 \\ \rho_1(u,\ v);\ (u,\ v) \in E_1 - E_2 \\ \rho_2(u,\ v);\ (u,\ v) \in E_2 - E_1 \end{cases}$$

**Example 12.** *Consider the two product fuzzy graphs $G_1 = (V_1, \mu_1,\ \rho_1)$ and $G_2 = (V_2, \mu_2,\ \rho_2)$ are given in Figure (3.4.a), (3.4.b) respectively. where $V_1 = \{u_1,\ u_2,\ u_3, u_5,\ u_6\}$, $\mu_1(u_6) = 0.4,\ \mu_1(u_1) = 0.1,\ \mu_1(u_2) = 0.3,\ \mu_1(u_3) = 0.2,\ \mu_1(u_5) = 0.6,$ $\rho(u,\ v) = \mu_1(u) \times \mu_1(v)$ for all $u, v \in E_1$, $V_2 = \{u_1,\ u_2,\ u_3,\ u_5,\ u_4\}$, $\mu_2(u_1) = 0.2,\ \mu_2(u_2) = 0.1,\ \mu_2(u_3) = 0.2,\ \mu_2(u_4) = 0.5,\ \mu_2(u_5) = 0.6$ and $\rho(u,\ v) = \mu_2(u) \times \mu_2(v)$ for all $u, v \in E_2$. We see that the union of $G_1$ and $G_2$ is a graph $G = G_1 \cup G_2 = (V, \mu,\ \rho)$ given in Figure (3.4.c), where $V = \{u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6\}$, $\mu(u_6) = 0.4,\ \mu(u_1) = 0.2,\ \mu(u_2) = 0.3,\ \mu(u_3) = 0.2,\ \mu(u_5) = 0.6,\ \mu(u_4) = 0.5,\ \rho(u_6,\ u_1) = 0.04,\ \rho(u_1,\ u_2) = 0.03,\ \rho(u_2,\ u_3) = 0.06,\ \rho(u_3,\ u_5) = 0.12, \rho(u_6,\ u_5) = 0.24$ and $\rho(u_4,\ u_5) = 0.3$.*

$G_1$ :

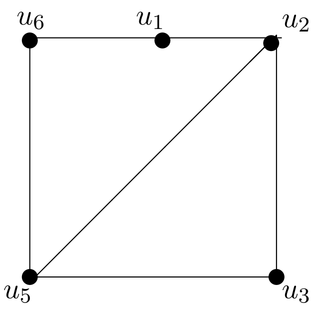



*Fig.* (3.4.*a*)

$G_2$ :

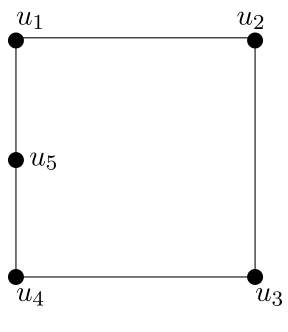
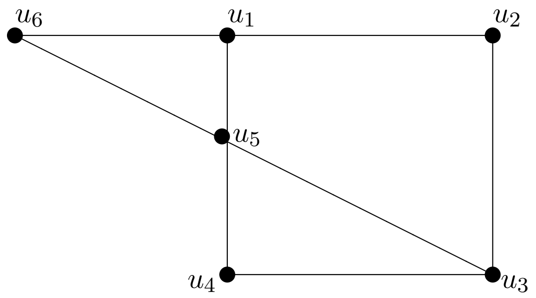



*Fig.* (3.4.*b*)

$G_1 \cup G_2$:




$Fig.$ (3.4.$c$)

**Theorem 13.** *Let* $G_1 = (\mu_1, \rho_1)$ *and* $G_2 = (\mu_2, \rho_2)$ *be two product fuzzy graph then,*
$(A)$ $\overline{(G_1 + G_2)} = \overline{G_1} \cup \overline{G_2}$
$(B)\overline{(G_1 \cup G_2)} = \overline{G_1} + \overline{G_2}$

*Proof.* Consider the identity map $I : V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove $(i)$ it is enough to prove
$A)(i)$ $(\overline{\mu_1 + \mu_2})(v_i) = \overline{\mu_1} \cup (\overline{\mu_2})(v_i)$,
$A)(ii)$ $(\overline{\rho_1 + \rho_2})(v_i, v_j) = \overline{\rho_1} \cup \overline{\rho_2}(v_i, v_j)$
Now, $A)(i)(\overline{\mu_1 + \mu_2})(v_i) = (\mu_1 + \mu_2)(v_i) =$

$$= \begin{cases} \mu_1(v_i); & v_i \in V_1 \\ \mu_2(v_i); & v_i \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_1}(v_i); & v_i \in V_1 \\ \overline{\mu_2}(v_i); & v_i \in V_2 \end{cases}$$

$$= (\overline{\mu_1} \cup \overline{\mu_2})(v_i),$$

$A)(ii)$ $(\overline{\rho_1 + \rho_1})(v_i, v_j) = (\mu_1 + \mu_2)(v_i) \times (\mu_1 + \mu_2(v_j) - (\rho_1 + \rho_2)(v_i, v_j)$

$$= \begin{cases} \mu_1(v_i) \times \mu_1(v_j) - \rho_1(v_i,\ v_j) \ \ if \ \ (v_i,\ v_j) \in E_1 \\ \mu_1(v_i) \times \mu_1(v_j) - \rho_2(v_i,\ v_j) \ \ if \ \ (v_i,\ v_j) \in E_2 \\ \mu_1(v_i) \times \mu_2(v_j) - \mu_1(v_i) \times \mu_2(v_j) \ \ if \ \ (v_i,\ v_j) \in E' \end{cases}$$

$$= \begin{cases} \overline{\rho_1}(v_i,\ v_j) \ \ if \ \ (v_i,\ v_j) \in E_1 \\ \overline{\rho_2}(v_i,\ v_j) \ \ if \ \ (v_i,\ v_j) \in E_2 \\ 0 \ \ \ if \ \ (v_i,\ v_j) \in E' \end{cases}$$

$$= (\overline{\rho_1} \cup \overline{\rho_2})(v_i,\ v_j)$$

□

Consider the identity map $I : V_1 \cup V_2 \longrightarrow V_1 \cup V_2$. To prove $(B)$, it is enough to prove
$B)(i)$ $(\overline{\mu_1 \cup \mu_2})(v_i) = \overline{\mu_1} + \overline{\mu}_2(v_i)$, $B)(ii)$ $(\overline{\rho_1 \cup \rho_2})(v_i,\ v_j) = \overline{\rho_1} \cup \overline{\rho_2}(v_i,\ v_j)$.

Now, $B(i)$ $(\overline{\mu_1 \cup \mu_2})(v_i) = (\mu_1 \cup \mu_2)(v_i)$

$$= \begin{cases} \mu_1(v_i); & v_i \in V_1 \\ \mu_2(v_i); & v_i \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_1}(v_i); & v_i \in V_1 \\ \overline{\mu_2}(v_i); & v_i \in V_2 \end{cases}$$

$$(\overline{\mu_1} \cup \overline{\mu_2})(v_i) = (\overline{\mu_1} + \overline{\mu_2})(v_i)$$

$B)(ii)$ $(\overline{\rho_1 \cup \rho_1})(v_i,\ v_j) = (\mu_1 \cup \mu_2)(v_i) \times (\mu_1 \cup \mu_2(v_j) - (\rho_1 \cup \rho_2)(v_i,\ v_j)$

$$= \begin{cases} \mu_1(v_i) \times \mu_1(v_j) - \rho_1(v_iv_j) \ \ if (v_i,\ v_j) \in E_1 \\ \mu_1(v_i) \times \mu_1(v_j) - \rho_2(v_iv_j) \ if \ (v_i,\ v_j) \in E_2 \\ \mu_1(v_i) \times \mu_2(v_j) - \mu_1(v_i) \times \mu_2(v_j) \ \ if \ \ (v_i,\ v_j) \in E' \end{cases}$$

$$= \begin{cases} \overline{\rho_1}(v_i,\ v_j) \ \ if \ \ (v_i,\ v_j) \in E_1 \\ \overline{\rho_2}(v_i,\ v_j)) \ \ if \ \ (v_i,\ v_j) \in E_2 \\ 0 \ \ if \ \ (v_i,\ v_j) \in E' \end{cases}$$

$$= (\overline{\rho_1} \cup \overline{\rho_2})(v_i,\ v_j) = (\overline{\rho_1} + \overline{\rho_2})(v_i,\ v_j).$$

In the following theorem we give $\gamma$ of the union of any not disjoint product fuzzy graphs $G_1$ and $G_2$, the relation between $\gamma$, of union and join and $\gamma_g$ of union and join respectively.

**Theorem 14.** *If $G_1$ and $G_2$ be any two not disjoint product fuzzy graphs, then*

$$\gamma(G_1 \cup G_2) = max(\gamma(G_1), \gamma(G_2)).$$

*Proof.* Let $D_1$ be a $\gamma - set$ of a product fuzzy graph $G_1$ and let $D_2$ be a $\gamma - set$ of a product fuzzy graph $G_2$. Then $D_1 \cup D_2$ is a dominating set of $G_1 \cup G_2$. Since $G_1$ and $G_2$ are not disjoint then $D_1 \cup D_2 \neq \phi$. Hence $\gamma(G_1 \cup G_2) = |D_1 \cup D_2| = max(\gamma(G_1), \gamma(G_2))$.

□

**Theorem 15.** *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ be two Product fuzzy graph such that $\rho_1(u,\ v) = \mu_1(u) \times \mu_1(v)$ for all $(u,\ v) \in E_1$, $\rho_2(u,\ v) = \mu_2(u) \times \mu_2(v)$, for all $(u,\ v) \in E_2$ then,*

$$\gamma(G_1 \cup G_2) = \gamma(G_1) + \gamma(G_2)$$

*Proof.* Let $D_1$ be a $\gamma_1$-set of a product fuzzy graph $G_1$ and $D_2$ be a $\gamma_2$-set of a product fuzzy graph $G_2$. Then $D_1 \cup D_2$ is a dominating set of $G_1 \cup G_2$. Hence $\gamma(G_1 \cup G_2) = |D_1 \cup D_2| = \gamma(G_1) + \gamma(G_2)$ □

**Theorem 16.** *Let $G_1 = (\mu_1,\ \rho_1)$ and $G_2 = (\mu_2,\ \rho_2)$ be two Product fuzzy graphs such that $\rho_1(u,v) = \mu_1(u) \times \mu_2(v)$ for all $(u,v) \in E_1$, $\rho_2(u,v) = \mu_2(u) \times \mu_2(v)$, for all $(u,v) \in E_2$ , then*

$$\gamma_g(G_1 \cup G_2) = \gamma_g(G_1) + \gamma_g(G_2).$$

*Proof.* Let $D_1$ be a $\gamma_g - set$ of a product fuzzy graph $G_1$ and $D_2$ be a $\gamma_g - set$ of a product fuzzy graph $G_2$. Then $D_1 \cup D_2$ is a global dominating set of $G_1 \cup G_2$. Hence $\gamma_g(G_1 \cup G_2) = |D_1 \cup D_2| = \gamma_g(G_1) + \gamma_g(G_2)$. □

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