An Approximate Solution of Higher Order Ordinary Differential Equations Using Modified Adomian Decomposition Method

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Abstract

Modified Adomian Decomposition Method (MADM) is used in this article to solve higher order ordinary differential equations. Some examples are proposed to show the ability of the methed for solving this type of equations.

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1 Introduction

Consider the frist order ordinary differential equation [6]

$$y' + P(x)y + f(x,y) = g(x), \tag{1}$$

with boundary condition $y(0) = A$. Where A is constant, $P(x)$ and $g(x)$ are given functions and $f(x,y)$ is real function.
It is interesting to note that the eq.(1) was derived by the

$$e^{-\int P(x)dx}\frac{d}{dx}e^{\int P(x)dx}(.) + f(x,y) = g(x). \tag{2}$$

Adomian decomposition method (ADM) [3-5] is a numerical method for solving ordinary and partial nonlinear differential equations. This method was improved from the 1970 to the 1990s by George Adomian, chair of the center for Applied Mathematics at the university of Georgia. The ADM has been successfully applied to solve nonlinear equation in studying many interesting problems arising in applied sciences and engineering [7,8]. We aim in this article to solving higher order ordinary differential equations. We submit a new modified through which we can solve ordinary differential equations of different order with high efficiency.

2 Building Ordinary Differential Equations of Higher Order

To derive the ordinary differential equations of different order, we use eq.(2) and put

$$\frac{d^n}{dx^n}e^{-\int P(x)dx}\frac{d}{dx}e^{\int P(x)dx}(.)+f(x,y)=g(x), \tag{3}$$

where $n \in N$. To determine such different equations of higher order we set n to different values.

1. Placing n=0 in the eq.(3) gives us the first order ordinary differential equation

$$y'+p(x)y+f(x,y)=g(x),$$

2. Placing n=1 in the eq.(3) gives us the second order ordinary differential equation [9]

$$y''+p(x)y'+p'(x)y+f(x,y)=g(x),$$

3. Placing n=2 in the eq.(3) gives us the third order ordinary differential equation

$$y'''+p(x)y''+2p'(x)y'+p''(x)y+f(x,y)=g(x).$$

Continuing with the same procedure until n gives us the following generalization:

$$y^{(n+1)}+\sum_{r=0}^{n}\binom{n}{r}p^{(r)}(x)y^{(n-r)}+f(x,y)=g(x). \tag{4}$$

3 The Adomian Decomposition Method

Consider the higher order ordinary differential equations in the form eq.(4)

$$Ly = g(x) - f(x,y), \tag{5}$$

where

$$L(.) = \frac{d^n}{dx^n} e^{-\int P(x)dx} \frac{d}{dx} e^{\int P(x)dx}(.) + f(x,y) = g(x), \tag{6}$$

and

$$L^{-1}(.) = e^{-\int P(x)dx} \int_0^x e^{\int P(x)dx} \underbrace{\int_0^x \int_0^x \int_0^x \ldots \int_0^x}_{(n)} (.) \underbrace{dxdxdxdx \ldots dx}_{(n+1)}.$$

By applying L^{-1} on (5), we have

$$y(x) = \beta(x) + L^{-1}g(x) - L^{-1}(f(x,y), \tag{7}$$

such that

$$L(\beta(x)) = 0.$$

The method by Adomian is given the solution $y(x)$ and the function $f(x,y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8}$$

and

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \tag{9}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen [1,2] to formulate Adomian polynomials. The following algorithm:

$$A_0 = F(y_0), A_1 = y_1 F'(y_0), A_2 = y_2' F(y_0) + \frac{1}{2!} y_1^2 F''(y_0),$$
$$A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + \frac{1}{3!} y_1^3 F'''(y_0), \tag{10}$$
$$\ldots.$$

Can be used to build Adomian polynomials, when $F(y)$ is any function. From (7),(8) and (9) we have

$$\sum_{n=0}^{\infty} y_n(x) = \beta(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{11}$$

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$\begin{aligned} y_0 &= \beta(x) + L^{-1}g(x), \\ y_{(n+1)} &= -L^{-1}A_n, \ n \geq 0, \end{aligned} \tag{12}$$

thus

$$y_0 = \beta(x) + L^{-1}g(x), y_1 = L^{-1}A_0, y_2 = L^{-1}A_1, y_3 = L^{-1}A_2 .... \tag{13}$$

From (10) and (13), we can determine the components y_n, and hence the series solution of $y(x)$ in (8) can be immediately obtained.

4 Numerical Examples

Example 1. First, let us consider the third order ordinary differential equation

$$y''' - 3x^2y'' - 12xy' - 6y = 6\left(-1 + 4\,x + 2\,x^2 - 4\,x^3 + 3\,x^4\right)y^4, \tag{14}$$

$$y(0) = 1, y'(0) = 0, y''(0) = -2,$$

obtained by placing $n = 2$, and $P(x) = -3x^2$, in eq. (4). We put

$$L(.) = \frac{d^2}{dx^2} e^{x^3} \frac{d}{dx} e^{-x^3}(.), \tag{15}$$

so

$$L^{-1}(.) = e^{x^3}\int_0^x e^{-x^3}\int_0^x\int_0^x (.). \tag{16}$$

Rewrite eq.(14) in an operator form

$$Ly = 6\left(-1 + 4\,x + 2\,x^2 - 4\,x^3 + 3\,x^4\right)y^4, \tag{17}$$

applying eq.(16) on both side of eq.(17) we get:

$$y(x) = 1 - x^2 + x^3 - \frac{3\,x^5}{5} + \frac{x^6}{2} - \frac{9\,x^8}{40} + \frac{x^9}{6} +$$
$$... + L^{-1}6\left(-1 + 4\,x + 2\,x^2 - 4\,x^3 + 3\,x^4\right)y^4, \tag{18}$$

replace the decomposition series $y_n(x)$ for $y(x)$ into (18) gives

$$\sum_{n=0}^{\infty} y_n(x) = 1 - x^2 + x^3 - \frac{3\,x^5}{5} + \frac{x^6}{2} - \frac{9\,x^8}{40} + \frac{x^9}{6} +$$
$$... + L^{-1}6\left(-1 + 4\,x + 2\,x^2 - 4\,x^3 + 3\,x^4\right)A_n), \tag{19}$$
$$y_0 = 1 - x^2 + x^3 - \frac{3\,x^5}{5} + \frac{x^6}{2} - \frac{9\,x^8}{40} + \frac{x^9}{6} + ...,$$
$$y_{n+1} = L^{-1}(A_n), \;\; n \geq 0, \tag{20}$$

by using Taylor series and Adomain polynomials in eq.(10) we obtain

$$y_0 = 1 - x^2 + x^3 - \frac{3\,x^5}{5} + \frac{x^6}{2} - \frac{9\,x^8}{40} + \frac{x^9}{6} + ...,$$
$$y_1 = -x^3 + x^4 + \frac{3\,x^5}{5} - \frac{17\,x^6}{10} + \frac{4\,x^7}{7} + \frac{75\,x^8}{56} - \frac{313\,x^9}{210} - \frac{41\,x^{10}}{210} + ...,$$
$$y_2 = \frac{x^6}{5} - \frac{4\,x^7}{7} - \frac{4\,x^8}{35} + \frac{59\,x^9}{42} - \frac{604\,x^{10}}{525} + ...,$$
$$y_3 = \frac{-17\,x^9}{210} + \frac{121\,x^{10}}{350} + ...$$

The series solution by (MADM) is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + ...,$$

that converges to the exact solution $y(x) = \frac{1}{1+x^2}$.

Example 2. Next, let us consider the fourth order ordinary differential equation

$$y'''' + \sqrt{1+x}y''' + \frac{3}{2\sqrt{1+x}}y'' - \frac{3}{4\sqrt[3]{1+x}}y' + \frac{3}{8\sqrt[5]{1+x}}y =$$

$$e^{y}-e^{x^{4}}+\frac{3\left(64\,x+240\,x^{2}+280\,x^{3}+105\,x^{4}+64\,(1+x)^{\frac{5}{2}}\right)}{8\,(1+x)^{\frac{5}{2}}}, \tag{21}$$

$$y(0)=y'(0)=y''(0)=y'''(0)=0,$$

obtained by placing $P(x)=\sqrt{1+x}, k=3$, in eq.(4) and $y(x)=x^4$ is the solution of eq.(21). Eq.(21) in an operator form becomes

$$Ly=e^{y}-e^{x^{4}}+\frac{3\left(64\,x+240\,x^{2}+280\,x^{3}+105\,x^{4}+64\,(1+x)^{\frac{5}{2}}\right)}{8\,(1+x)^{\frac{5}{2}}}, \tag{22}$$

where

$$L(.)=\frac{d^{3}}{dx^{3}}e^{\frac{-2\,(1+x)^{\frac{3}{2}}}{3}}\frac{d}{dx}e^{\frac{2\,(1+x)^{\frac{3}{2}}}{3}}(.) \tag{23}$$

and

$$L^{-1}(.)=e^{\frac{-2\,(1+x)^{\frac{3}{2}}}{3}}\int_0^x e^{\frac{2\,(1+x)^{\frac{3}{2}}}{3}}\int_0^x\int_0^x\int_0^x(.)dxdxdxdx, \tag{24}$$

applying eq.(25) on eq.(22) we find

$$y(x)=L^{-1}(e^{y})+\frac{23\,x^{4}}{24}+\frac{x^{5}}{120}+\frac{x^{6}}{480}-\frac{11\,x^{7}}{6720}-\frac{x^{8}}{15360}-\frac{x^{9}}{8960}+\frac{137\,x^{10}}{1075200}+...,$$

$$y_0=\frac{23\,x^{4}}{24}+\frac{x^{5}}{120}+\frac{x^{6}}{480}-\frac{11\,x^{7}}{6720}-\frac{x^{8}}{15360}-\frac{x^{9}}{8960}+\frac{137\,x^{10}}{1075200}+...,$$

$$y_{n+1}=L^{-1}(A_n),\ \ n\geq 0. \tag{25}$$

by using Taylor series with order 10 and eq.(10) we get,

$$y_0=\frac{23\,x^{4}}{24}+\frac{x^{5}}{120}+\frac{x^{6}}{480}-\frac{11\,x^{7}}{6720}-\frac{x^{8}}{15360}-\frac{x^{9}}{8960}+\frac{137\,x^{10}}{1075200}+...,$$

$$y_1=\frac{x^{4}}{24}-\frac{x^{5}}{120}-\frac{x^{6}}{480}+\frac{11\,x^{7}}{6720}+\frac{13\,x^{8}}{322560}+\frac{17\,x^{9}}{145152}-\frac{3667\,x^{10}}{29030400}+...,$$

$$y_2=\frac{x^{8}}{40320}-\frac{x^{9}}{181440}-\frac{x^{10}}{907200}+....$$

$$y(x)=y_0+y_1+y_2=x^4.$$

Example 3. Finally, let us consider the sixth order ordinary differential equation

$$y^{(6)}+\ sinxy^{(5)}+5\ cosxy^{(4)}-10\ sinxy^{(3)}-10\ cosxy''+5\ sinxy'+\ cosxy=$$

$$x\left(5040 - x^{13} + x^2\left(4200 - 420\,x^2 + x^4\right)\cos x\right) \\ +35\,x^2\left(72 - 60\,x^2 + x^4\right)\sin x + y^2, \qquad (26)$$

with

$$y(0) = y'(0) = y''(0) = y'''(0) = y''''(0) = y'''''(0) = 0,$$

obtained by placing $P(x) = \sin x, k = 5$, in eq.(4) and $y(x) = x^7$ is the solution of eq.(26). Eq.(26) in an operator form becomes

$$Ly = x\left(5040 - x^{13} + x^2\left(4200 - 420\,x^2 + x^4\right)\cos x\right) \\ +35\,x^2\left(72 - 60\,x^2 + x^4\right)\sin x + y^2, \qquad (27)$$

where

$$L(.) = \frac{d^5}{dx^5}e^{\cos x}\frac{d}{dx}e^{-\cos x}(.), \qquad (28)$$

and

$$L^{-1}(.) = e^{\cos x}\int_0^x e^{-\cos x}\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x(.)dxdxdxdxdxdx, \qquad (29)$$

applying eq.(29) on eq.(27) we find

$$y(x) = x^7 - \frac{x^{20}}{27907200} + ... + L^{-1}(y^2),$$

$$y_0 = x^7 - \frac{x^{20}}{27907200} + ...,$$
$$y_{n+1} = L^{-1}(A_n), \;\; n \geq 0. \qquad (30)$$

By using Taylor series and eq.(10) we get,

$$y_0 = x^7 - \frac{x^{20}}{27907200} + ..., y_1 = \frac{x^{20}}{27907200} + ...$$
$$y(x) = y_0 + y_1 = x^7.$$

Conclusion

The MADM that introduced in this article and the results obtained from the three examples have shown that MADM is more a powerful and easy technique in finding an approximate solutions. As we noted in the frist example, the result was very close to the exact solution. In examples 2,3 we obtained the exact solution.

References

[1] A. M. Wazaz, A First Course in Integral Equation, World Scientic, Singapore, 1997.

[2] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (1) (2000) 53-69.

[3] G. Adomian, A review of the decomposition method and some recent results for nonlinear equation, Math. Comput. Modelling 13(7)(1992) 17-43.

[4] G. Adomian, Solving forontier problems of physics: The Decomposition Method, Kluwer, Boston, MA, 1994.

[5] G. Adomian, and R.Rach, Inversion of nonlinear stochastic operators," J.Math. Anal. Appl.91(1983) 39-46.

[6] Y. Q. Hasan, Solving first order ordinary differential equation by modified Adomian decomposition method, (AITS), 1 (4) (2012), 86-89.

[7] E. Babolian, A. R. Vahidi, G. H. Asadi Cordshooli, Solving differential equations by decomposition method, Appl. Math. Comput. 167 (2) (2005) 1150-1155.

[8] H. Bayramoglu, N. Bildik, The solution of two dimensional nonlinear differential equation by the Adomian decomposition method, Appl. Math. Comput. 163 (2) (2005) 519-524.

[9] N.M. Dabwan and Y. Q. Hasan, Solving Second Order Ordinary Differential Equation Using a New Modified Adomian Method, AMSJ 9(3)(2020) 937-943.