FRACTIONAL VAN DER POL EQUATION BY ADOMIAN DECOMPOSITION METHOD

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Abstract

Adomian Decomposition Method (ADM) is successfully used to find the approximate solution of fractional Van der Pol equation. The ADM can be used to solve the ordinary, partial and fractinal differential equations. In this work, we will study fractional Van der Pol equation by (ADM). The numerical resultes of $y(x)$ for the considered fractional Van der Pol equation is obtained.

Keywords : Adomian Decomposition Method; Fractional Van der Pol Equation; Initial Conditions.

1. Introduction

The Dutch physicit Van der Pol had formulated one of the most well know equations, that is the equation of non-linear dynamics. In the begining, such equation was used as a model for an electrical circuit with a triode value. In the later stages, it was widely investigated as a host of a rich class of dynamical behavior [3]. The Van der Pol oscillator is regarded as one of the most well know examples of selfoscillatory systems. Therefore, it is now taken as a very useful mathematical model that can be applied to various types of modified systems. fractional differential equations have generated much interest to the scientists in different fields like biology, physics and electrical networks [7].
In 1980s Adomian discovered a new technique for solving linear and nonlinear ordinary and partial differential equations, algebraic equations, fractinal

equations, integral differential. He called the technique is Adomian Decomposition Method (ADM). This method is very useful and successful to solve these types of equations and the convergence analysis of the ADM was discussed in [1]. Y. Cherru ault and G. Adomian give the new proof of convergence analysis of the decomposition method [2]. Study of fractional order Van der Pol equation [6]. In [5] S. G. Othman and Y. Q. Hasan solve and studied the approximate solution for Ven der Pol equation via (ADM), and they found that the results are good, accurate and convergence of the exact solution. After that many modifications which made by numerous researchers in an attempt to improve the accuracy of this method. This work is sutying fractional Van der Pol equation.
For this reason, we give reliable differential operators of fractional Van der Pol equation via (ADM).

2. Adomian Decomposition Method for fractional Van der Pol equation

The fractional Van der Pol's equation take the following shape:

$$D_x^{2\alpha} y - \mu(1-y^2)D_x^{\alpha} y + \alpha^2 y = g(x), \tag{1}$$

with the following initial conditions

$$y(0) = A, D_x^{\alpha} y(0) = B,$$

where μ is seclar parameter, g(x) is given function, $0 < \alpha \leq 1$.
Take the differential operator L in the term of $D_x^{2\alpha} y + \alpha^2 y$, then Eq.(1) can write it as below :

$$L_\alpha y = g(x) + \mu(1-y^2)D_x^{\alpha} y. \tag{2}$$

We give differential operators L as below:

$$L_\alpha(.) = e^{-ix^\alpha} \frac{d^\alpha}{dx^\alpha} e^{2ix^\alpha} \frac{d^\alpha}{dx^\alpha} e^{-ix^\alpha}(.), \tag{3}$$

$$L_\alpha(.) = \frac{1}{\cos x^\alpha} \frac{d^\alpha}{dx^\alpha} \cos^2 x^\alpha \frac{d^\alpha}{dx^\alpha} \frac{1}{\cos x^\alpha}(.), \tag{4}$$

The invers differential operators L_α^{-1} are defined respectively as

$$L_\alpha^{-1}(.) = e^{ix^\alpha} \int_0^x e^{-2ix^\alpha} \int_0^x e^{ix^\alpha}(.)dx^\alpha dx^\alpha, \tag{5}$$

$$L_\alpha^{-1}(.) = \cos x^\alpha \int_0^x \cos^{-2} x^\alpha \int_0^x \cos x^\alpha (.)dx^\alpha dx^\alpha, \tag{6}$$

Operating with inverse differential operators (5) and (6) on (2), and using the initial conditions $y(0) = A$, $D_x^\alpha y(0) = B$, we get

$$y(x) = Acosx^\alpha + \frac{B}{\alpha} sinx^\alpha + L_\alpha^{-1} g(x) + \mu L_\alpha^{-1}(1 - y^2) D_x^\alpha y. \tag{7}$$

According to the ADM, the solution $y(x)$ is represented by the decomposition series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{8}$$

and the non-linear term [14], for equation (1), shown by the decomposition series

$$N(y(x)) = \mu(1 - y_n^2) y_n' = \mu(1 - A_n) D_x^\alpha y_n, \tag{9}$$

where $A_n(x)$, the Adomian polynomials, defined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \Big[N(\sum_{i=0}^{\infty} \lambda^i y_i) \Big]_{\lambda=0}, n = 0, 1, 2, ...$$

Substituing (12) and (13) in (10), we have

$$\sum_{n=0}^{\infty} y_n(x) = Acosx^\alpha + \frac{B}{\alpha} sinx^\alpha + L_\alpha^{-1}(g(x) + \mu L_\alpha^{-1} \sum_{n=0}^{\infty} (1 - A_n) D_x^\alpha y_n. \tag{10}$$

Can write recursive formula as

$$y_0(x) = Acosx^\alpha + \frac{B}{\alpha} sinx^\alpha + L_\alpha^{-1} g(x), \tag{11}$$

and

$$y_{n+1}(x) = \mu L_\alpha^{-1}(1 - A_n) D_x^\alpha y_n, n \geq 0, \tag{12}$$

For numerical purposes, the n-part approximant

$$\phi_n(x) = \sum_{i=0}^{n-1} y_i,$$

with

$$y(x) = \lim_{n \to \infty} \phi_n(x).$$

3. Numerical Applications

In this part, two examples for solving fractional Van der Pol equation using ADM .

Example 1. Assume the problem with initial values

$$D_x^{2\alpha} y - \mu(1-y^2) D_x^{\alpha} y + \alpha^2 y = 2\alpha^2 e^{x^\alpha}, \tag{13}$$

$$y(0) = 1, D_x^{\alpha} y(0) = \alpha,$$

with exact solution $y(x) = e^{x^\alpha}$, put $\mu = 0$.
In an operator form (4), equation (13) becomes

$$L_\alpha y = 2\alpha^2 e^{x^\alpha}, \tag{14}$$

take inverse differential operator L_α^{-1} to both side of (14), we get

$$y(x) = y(0) cos x^\alpha + \frac{D_x^\alpha y(0)}{\alpha} sin x^\alpha + L_\alpha^{-1}(2\alpha^2 e^{x^\alpha}), \tag{15}$$

under the initial conditions $y(0) = 1$, $D_x^\alpha y(0) = \alpha$, the solution yields

$$y(x) = e^{x^\alpha}.$$

Example 2. Consider the non-linear fractional Van der Pol equation

$$D_x^{2\alpha} y - \mu(1-y^2) D_x^\alpha y + \alpha^2 y = -2\, x^{5\alpha}\, \alpha + 2\, x^\alpha\, \alpha + 2\, \alpha^2 + x^{2\,\alpha}\, \alpha^2 - \mu(1-y^2) D_x^\alpha y, \tag{16}$$

$$y(0) = D_x^\alpha y(0) = 0,$$

with exact solution $y(x) = x^{2\alpha}$, put $\mu = 1$.

In an operator form (3), equation (16) becomes

$$L_\alpha y = -2\, x^{5\alpha}\, \alpha + 2\, x^\alpha\, \alpha + 2\, \alpha^2 + x^{2\,\alpha}\, \alpha^2 - (1-y^2) D_x^\alpha y, \tag{17}$$

to take inverse differential operator L_α^{-1} to both side of (17), with initial conditions $y(0) = D_x^\alpha y(0) = 0$, the solution yields

$$y(x) = L_\alpha^{-1}(-2\, x^{5\alpha}\, \alpha + 2\, x^\alpha\, \alpha + 2\, \alpha^2 + x^{2\,\alpha}\, \alpha^2) - L_\alpha^{-1}(1-y^2) D_x^\alpha y,$$

whereas

$$y_0 = 0 + L_\alpha^{-1}(-2\, x^{5\alpha}\, \alpha + 2\, x^\alpha\, \alpha + 2\, \alpha^2 + x^{2\,\alpha}\, \alpha^2),$$

and

$$y_{n+1} = -L_\alpha^{-1}(1-y_n^2) D_x^\alpha y_n, \tag{18}$$

where the polynomials Adomian for $f(y) = y^2$ are

$$A_0 = f(y_0) = y_0^2,$$

$$A_1 = y_1 f'(y_0) = 2y_0 y_1, n \geq 0$$

so

$$y_0 = x^{2\,\alpha} + \frac{x^{3\,\alpha}}{3\,\alpha} - \frac{x^{5\,\alpha}}{60\,\alpha} - \frac{17\,x^{7\,\alpha}}{360\,\alpha} + \frac{17\,x^{9\,\alpha}}{25920\,\alpha},$$

$$y_1 = \frac{-x^{4\,\alpha}}{12\,\alpha^3} + \frac{x^{6\,\alpha}}{180\,\alpha^3} + \frac{319\,x^{8\,\alpha}}{6720\,\alpha^3} - \frac{x^{3\,\alpha}}{3\,\alpha^2} + \frac{x^{5\,\alpha}}{60\,\alpha^2} + \frac{17\,x^{7\,\alpha}}{360\,\alpha^2},$$

$$y_2 = x^{9\,\alpha}\left(\frac{-1}{2592\,\alpha^8} + \frac{151}{15120\,\alpha^6}\right) - \frac{x^{8\,\alpha}}{288\,\alpha^7} - \frac{x^{7\,\alpha}}{126\,\alpha^6},$$

$$\ldots$$

Table 1. Difference between ADM and Exact solution $y(x) = x^{2\alpha}$.

x	Exac solution when $\alpha = 0.9$	ADM when $\alpha = 0.9$	Absolute Error	Exac solution when $\alpha = 0.8$	ADM when $\alpha = 0.8$	Absolute Error
0.0	0.00	0.00	0.00	0.00	0.00	0.00
0.1	0.0157382	0.0158489	0.000110	0.0246022	0.0251189	0.0005167
0.2	0.0543105	0.0551892	0.0008787	0.0730282	0.0761462	0.003118
0.3	0.111438	0.114503	0.003065	0.136537	0.145678	0.009141
0.4	0.184655	0.19218	0.007525	0.21111	0.230832	0.019722

From teble 1. The difference in the solution between the ADM $y(x) = \sum_{n=0}^{2} y_i(x)$ and Exact solution $y(x) = x^{2\alpha}$. We note that, the solution by this method is converge.

4. Conclusion

In this work, we gave us differentail operators to solve fractional Van der Pol equation by Adomian Decomposition Method (ADM). (ADM) is effective method for solving the non-linear differential equations. The Van der Pol equation is a model of non-linear differential equations. We gave us two examples, the solutions we got by these differential operators are more accurate and faster convergence of the exact solution.

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