Matrix operators on the new spaces of λ-difference sequences

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Abstract. In the present paper, we have concluded the α-, β- and γ-duals of the λ-difference spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ which have recently been introduced. Further, we have characterized the matrix operators acting on, into and between these new sequence spaces.

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1 Introduction

By w, we denote the linear space of all real or complex sequences with coordinate-wise addition and scalar multiplication (over the scalar field of real or complex numbers). If $x \in w$; we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^\infty$ and we will use the conventions that $|x| = (|x_k|)$ and $xy = (x_k\, y_k)$ for $x, y \in w$, and $x/y = (x_k/y_k)$ as well as $1/y = (1/y_k)$, where $y_k \neq 0$ for all $k \geq 1$. Also, any term with a non-positive subscript is equal to naught (e.g., the terms x_0 and y_{-1} are meaningless). Any vector subspace of w is called a *sequence space*, and a normed sequence space is called a *BK-space* if it is complete and its coordinate-maps are all continuous. That is, a BK-space is a Banach sequence space with continuous coordinates. For instance, the classical sequence spaces ℓ_∞, c, c_0 and ℓ_p $(p \geq 1)$ are BK-spaces with their usual norms, where ℓ_∞, c and c_0 are respectively the spaces of all bounded, convergent and null sequences, normed by $\|x\|_\infty = \sup_k |x_k|$ (the supremum is taken over all integers $k \geq 1$) and the space ℓ_p (for each real $p \geq 1$) is consisting of all sequences associated with p-absolutely convergent series with p-norm $\|x\|_p = (\sum_{k=1}^\infty |x_k|^p)^{1/p}$. Further, we write bv_p $(p \geq 1)$ for the space of all sequences with p-bounded variation, that is $bv_p = \{x \in w : (x_k - x_{k-1}) \in \ell_p\}$. Moreover, by bs, cs and cs_0, we mean the spaces of all sequences associated with bounded, convergent and null series, respectively. That is $bs = \{x \in w : (\sum_{k=1}^n x_k) \in \ell_\infty\}$, $cs = \{x \in w : (\sum_{k=1}^n x_k) \in c\}$ and $cs_0 = \{x \in w : (\sum_{k=1}^n x_k) \in c_0\}$. For a given sequence space X, the α-, β- and γ-*duals* of X are respectively denoted by X^α, X^β and X^γ, which can be defined as follows:

$$X^\alpha = \Big\{a \in w : \ ax = (a_k\, x_k) \in \ell_1 \text{ for all } x \in X\Big\},$$
$$X^\beta = \Big\{a \in w : \ ax = (a_k\, x_k) \in cs \text{ for all } x \in X\Big\},$$
$$X^\gamma = \Big\{a \in w : \ ax = (a_k\, x_k) \in bs \text{ for all } x \in X\Big\}.$$

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1

Besides, it is well-known that $X^\alpha \subset X^\beta \subset X^\gamma$, the inclusion $X \subset Y$ implies that $Y^\theta \subset X^\theta$, and we have $c_0^\theta = c^\theta = \ell_\infty^\theta = \ell_1$, $\ell_1^\theta = \ell_\infty$ and $\ell_p^\theta = \ell_q$ for $p > 1$ with $q = p/(p-1)$, where $\theta = \alpha, \beta$ or γ. Due to the infinite dimensions of sequence spaces in the general case, the notion of *matrix transformations* between sequence spaces has been arisen for study the linear operators between such spaces. For an infinite matrix A with real or complex entries a_{nk} $(n, k \geq 1)$, we write $A = [a_{nk}]_{n,k=1}^\infty$ or simply $A = [a_{nk}]$, and we will write A_n for the n-th row sequence in A, that is $A_n = (a_{nk})_{k=1}^\infty$ for each $n \geq 1$. Also, for any sequence $x \in w$, the A-*transform* of x, denoted by $A(x)$, is defined to be the sequence $A(x) = (A_n(x))_{n=1}^\infty$ whose terms given by

$$A_n(x) = \sum\nolimits_{k=1}^\infty a_{nk}\, x_k \qquad (n \geq 1)$$

provided convergence of series for each $n \geq 1$ and we then say that $A(x)$ exists. Further, for any two sequence spaces X and Y, we say that A acts from X into Y if $A(x)$ exists and $A(x) \in Y$ for every $x \in X$. Furthermore, the *matrix class* (X, Y) is define to be the collection of all infinite matrices acting from X into Y. In fact, there may exists an infinite matrix A such that $A \notin (X, Y)$ and so the infinite matrices in the class (X, Y) must be characterized from those matrices which are not in (X, Y). That is, there must exist a list of necessary and sufficient conditions for a given infinite matrix A to be in (X, Y), where $A \in (X, Y)$ if and only if $A(x)$ exists as well as $A(x) \in Y$ for every $x \in X$. In other words, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for every $n \geq 1$ and $A(x) \in Y$ for all $x \in X$, where each A_n is the n-th row sequence in A, and so the β-duality is an important tool for characterizing matrix classes. Obviously, if $A \in (X, Y)$; then A defines a linear operator $A : X \to Y$ by $x \mapsto A(x)$, and we may call it as a *matrix operator* (matrix mapping) and the same for every linear operator from X into Y which can be given by an infinite matrix. Moreover, it is worth mentioning that the most general forms of linear operators between sequence spaces can be given by infinite matrices. This fact gives a special importance for the notion of matrix transformations between sequence spaces, which has been studied by several authors in many research papers (see [5, 7, 8, 13, 15, 16]) and has recently been used to introduce new sequence spaces and characterize their matrix classes by means of the idea of *matrix domains* (see [1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 17]). For an infinite matrix A and a sequence space X, the matrix domain of A in X is a sequence space X_A defined as follows:

$$X_A = \{x \in w : \ A(x) \in X\}.$$

The most useful cases of matrix domains are those obtained from a special type of infinite matrices called as *triangles*, where an infinite matrix $T = [t_{nk}]$ is called a triangle if $t_{nn} \neq 0$ for every $n \geq 1$ and $t_{nk} = 0$ for all $k > n$ $(n, k \geq 1)$. For example, the sum-matrix σ and the band-matrix Δ are infinite matrices which are triangles defining the partial sum and the difference operator, respectively. That is $\sigma(x) = (\sigma_n(x))$ and $\Delta(x) = (\Delta(x_n))$ such that $\sigma_n(x) = \sum_{k=1}^n x_k$ and $\Delta(x_n) = x_n - x_{n-1}$ for all $n \geq 1$, where $\Delta(x_1) = x_1$. This leads us to note that $cs_0 = (c_0)_\sigma$, $cs = (c)_\sigma$, $bs = (\ell_\infty)_\sigma$ and $bv_p = (\ell_p)_\Delta$ for $p \geq 1$. Similarly $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$, $c(\Delta) = (c)_\Delta$ and $c_0(\Delta) = (c_0)_\Delta$, where $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ are the *difference spaces* of bounded, convergent and null difference sequences, respectively. In this paper, we will deduce some important results concerning sereis and difference spaces. Further, we will conclude the α-, β- and

γ-duals of the λ-difference spaces $\ell_\infty(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $c_0(\Delta^\lambda)$, and characterize matrix operators acting on, into and between these new sequence spaces.

2 Notations and Preliminaries

In this section, we will display our notations and deduce some important lemmas about series which will be needed to prove our main results in next sections.

For simplicity in notation, we will use the symbol μ to stand for any one of the spaces c_0, c or ℓ_∞. Thus, by $\mu(\Delta)$ we mean the respective one of the spaces $c_0(\Delta)$, $c(\Delta)$ or $\ell_\infty(\Delta)$, and so the corresponding space of $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$ will be denoted by $\mu(\Delta^\lambda)$. Also, throughout this paper, we assume that $\lambda = (\lambda_k)_{k=1}^\infty$ is a strictly increasing sequence of positive reals, that is $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Then, the λ-*matrix* $\Lambda = [\lambda_{nk}]_{n,k=1}^\infty$ is a triangle defined by

$$\lambda_{nk} = \begin{cases} \dfrac{\Delta(\lambda_k)}{\lambda_n}\,; & (1 \le k \le n), \\ 0\,; & (k > n \ge 1). \end{cases} \tag{2.1}$$

Thus, for every sequence $x \in w$, we have

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n} \Delta(\lambda_k)\, x_k \qquad (n \ge 1). \tag{2.2}$$

The λ-*sequence spaces* μ^λ (c_0^λ, c^λ and ℓ_∞^λ) have been introduced in [11] and [12] as matrix domains of Λ in μ. That is, the spaces μ^λ are defined as follows:

$$\mu^\lambda = (\mu)_\Lambda = \big\{x \in w :\ \Lambda(x) \in \mu\big\}.$$

Further, we put $\tilde{\Lambda} = \Delta\Lambda$ and for every $x \in w$ we have

$$\tilde{\Lambda}_n(x) = \Delta(\Lambda_n(x)) = \Lambda_n(x) - \Lambda_{n-1}(x) \qquad (n \ge 1).$$

In other words, the infinite matrix $\tilde{\Lambda} = [\tilde{\lambda}_{nk}]_{n,k=1}^\infty$ is a triangle given by

$$\tilde{\lambda}_{nk} = \begin{cases} \Delta\Big(\dfrac{1}{\lambda_n}\Big)\,\Delta(\lambda_k)\,; & (1 \le k < n), \\ \dfrac{\Delta(\lambda_n)}{\lambda_n}\,; & (k = n), \\ 0\,; & (k > n \ge 1). \end{cases} \tag{2.3}$$

Then, for any $x \in w$, we find that

$$\tilde{\Lambda}_n(x) = \frac{\Delta(\lambda_n)}{\lambda_n}\, x_n + \Delta\Big(\frac{1}{\lambda_n}\Big)\sum_{k=0}^{n-1} \Delta(\lambda_k)\, x_k \qquad (n \ge 1), \tag{2.4}$$

where the summation is started from $k = 0$ to be suitable with the case $n = 1$. This can equivalently be written as follows ($\tilde{\Lambda}_1(x) = \Lambda_1(x) = x_1$):

$$\tilde{\Lambda}_n(x) = \Big(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\Big)\sum_{k=2}^{n} \lambda_{k-1}\, \Delta(x_k) \qquad (n > 1).$$

3

Moreover, the λ-*difference spaces* $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$ have recently been introduced in [14] as matrix domains of Λ in $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ respectively, which can also be defined as matrix domains of $\tilde{\Lambda}$ in c_0, c and ℓ_∞, respectively. That is $\mu(\Delta^\lambda)$ are sequence spaces defined as follows:

$$\mu(\Delta^\lambda) = (\mu)_{\tilde{\Lambda}} = \big\{x \in w : \ \tilde{\Lambda}(x) \in \mu\big\}$$

which are BK-spaces with norm $\|\cdot\|_{\Delta^\lambda}$ given by $\|x\|_{\Delta^\lambda} = \|\tilde{\Lambda}(x)\|_\infty$ for all $x \in \mu(\Delta^\lambda)$. Furthermore, the inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ are satisfied with equalities if and only if $\lambda/\Delta(\lambda) \in \ell_\infty$. Also, the inclusion $c(\Delta) \subset c(\Delta^\lambda)$ doesn't hold if $\Delta(\lambda/\Delta(\lambda))$ is oscillated. In fact, if either $\Delta(\lambda/\Delta(\lambda)) \in c$ or $\Delta(\lambda/\Delta(\lambda)) \to \infty$; then $c(\Delta) \subset c(\Delta^\lambda)$. That is, the inclusion $c(\Delta) \subset c(\Delta^\lambda)$ holds if and only if $\lambda/\Delta(\lambda) \in c(\Delta^\lambda)$, and the identity holds if and only if $\lambda/\Delta(\lambda) \in \ell_\infty \cap c_0(\Delta)$. We refer the reader to [14] for further knowledge about the spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ and $\ell_\infty(\Delta^\lambda)$.

Now, as a natural continuation of work done in [14], we will conclude the α-, β- and γ-duals of the λ-difference spaces $\mu(\Delta^\lambda)$ and characterize some related matrix classes. For this we will use the following notations and terminologies:

Here and in the sequel, we assume that $a \in cs$. Then, the series $\sum_{j=1}^\infty a_j$ converges and we denote its k-th remainder $\sum_{j=k}^\infty a_j$ by $R_k(a)$ or simply R_k for all $k \geq 1$, and so $R = (R_k)$ is the sequence of all those remainders. Also, the finite sum $\sum_{j=k}^n a_j$ will be denoted by $R_k^n(a)$ or simply R_k^n for all $n \geq 1$ and every $k \leq n$. That is, we have

$$R_k := R_k(a) = \sum_{j=k}^{\infty} a_j \quad \text{and} \quad R_k^n := R_k^n(a) = \sum_{j=k}^{n} a_j \qquad (k \geq 1, n \geq k). \tag{2.5}$$

Thus $R_k = \lim_{n\to\infty} R_k^n$ and $R_k = R_k^n + R_{n+1}$ which implies that $\big||R_k^n| - |R_k|\big| \leq |R_k^n - R_k| = |R_{n+1}|$ $(1 \leq k \leq n)$ and by applying the triangle inequality, we obtain that

$$\Big|\sum_{k=1}^n |R_k^n| - \sum_{k=1}^n |R_k|\Big| \leq n|R_{n+1}| \qquad (n \geq 1). \tag{2.6}$$

Also, for every $n \geq 1$, we have

$$\sum_{k=1}^n |a_k| = |a_n| + \sum_{k=1}^{n-1} |R_k^n - R_{k+1}^n| \leq \sum_{k=1}^n |R_k^n| + \sum_{k=2}^n |R_k^n|$$

and by adding $|R_1^n|$ to both sides, we deduce the following inequality:

$$\sum_{k=1}^n |a_k| \ \leq \ \sum_{k=1}^n |a_k| + \Big|\sum_{k=1}^n a_k\Big| \ \leq \ 2\sum_{k=1}^n |R_k^n| \qquad (n \geq 1). \tag{2.7}$$

Further, we will frequently use the following sum-formula:

$$\sum_{k=r}^n s_k \sum_{j=r}^k t_j = \sum_{k=r}^n t_k \sum_{j=k}^n s_j \qquad (1 \leq r \leq n)$$

which still valid if n is replaced by ∞ provided that series are convergent. Thus, with our assumption $a \in cs$, we obtain that

$$R_{n+1} \ = \ \sum_{m=n+1}^\infty \Big(\frac{1}{m} - \frac{1}{m+1}\Big) \sum_{j=n+1}^m j a_j \qquad (n \geq 1). \tag{2.8}$$

4

Moreover, for every $x \in w$, we have $x_k = \sum_{j=1}^{k} \Delta(x_j)$ for $k \geq 1$ and so it follows that

$$\sum\nolimits_{k=1}^{n} a_k x_k = \sum\nolimits_{k=1}^{n} R_k^n \, \Delta(x_k) = \sum\nolimits_{k=1}^{n} R_k \, \Delta(x_k) - x_n R_{n+1} \qquad (n \geq 1), \qquad (2.9)$$

$$\sum\nolimits_{k=1}^{n} k a_k = \sum\nolimits_{k=1}^{n} R_k^n = \sum\nolimits_{k=1}^{n} R_k - n R_{n+1} \qquad (n \geq 1), \qquad (2.10)$$

where (2.10) can also be obtained from (2.9) by taking $x_k = k$ for all $k \geq 1$.
In addition, let us consider the particular case $a \in \ell_1$ in which $|a| = (|a_k|) \in cs$ and so $a \in cs$. Then, all above relations are valid and we also define $\bar{R}_k^n = R_k^n(|a|)$ for $1 \leq k \leq n$, and $\bar{R} = (\bar{R}_k)$ is the sequence defined by $\bar{R}_k = R_k(|a|)$ for every $k \geq 1$. That is, we have the following notations:

$$\bar{R}_k := R_k(|a|) = \sum_{j=k}^{\infty} |a_j| \quad \text{and} \quad \bar{R}_k^n := R_k^n(|a|) = \sum_{j=k}^{n} |a_j| \qquad (k \geq 1, n \geq k). \quad (2.11)$$

Thus, from (2.10) with $|a|$ instead of a, we find that

$$\sum\nolimits_{k=1}^{n} |k a_k| = \sum\nolimits_{k=1}^{n} \bar{R}_k^n = \sum\nolimits_{k=1}^{n} \bar{R}_k - n \bar{R}_{n+1} \qquad (n \geq 1). \qquad (2.12)$$

Further, we have $\Delta(\sum_{k=1}^{n} \bar{R}_k^n) = n|a_n| \geq 0$ and $\bar{R}_k \geq 0$ as well as $\bar{R}_k^n \geq 0$ for every $k \geq 1$ and all $n \geq k$. Thus, we deduce some facts in the following Remark:

Remark 2-1 For every $a \in \ell_1$, we have the following:
(1) $\sum_{k=1}^{n} |\bar{R}_k| = \sum_{k=1}^{n} \bar{R}_k$ and $\sum_{k=1}^{n} |\bar{R}_k^n| = \sum_{k=1}^{n} \bar{R}_k^n$ for all n.
(2) $\bar{R} \in cs \iff \bar{R} \in \ell_1 \iff \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} |a_j| < \infty$.
(3) The sequence $(\sum_{k=1}^{n} \bar{R}_k^n)_{n=1}^{\infty}$ is increasing of non-negative reals.
(4) If $\bar{R} \in \ell_1$; then $(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty$, as $\sum_{k=1}^{n} \bar{R}_k^n \leq \sum_{k=1}^{n} \bar{R}_k \leq \sum_{k=1}^{\infty} \bar{R}_k$ $(n \geq 1)$.

Remark 2-2 We have the following true implications:
(1) $(n a_n) \in bs \Longrightarrow a \in cs$. Also $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow a \in cs$.
(2) $(n a_n) \in cs \Longrightarrow a \in cs$. Also $(\sum_{k=1}^{n} R_k^n) \in c \Longrightarrow a \in cs$.
(3) $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty \Longrightarrow a \in \ell_1$. Also $(\sum_{k=1}^{n} |R_k^n|) \in c \Longrightarrow a \in \ell_1$.
(4) $(n a_n) \in \ell_1 \Longrightarrow a \in \ell_1$. Also $(\sum_{k=1}^{n} \bar{R}_k^n) \in \ell_\infty \Longrightarrow a \in \ell_1$.

Proof of Remark 2-2. Although the given implications are obviously true, we may give some justifications: For (1), assume $(n a_n) \in bs$, and since $(1/n) \in bv_0$; we deduce that $a = (a_n) = (1/n)(n a_n) \in cs$, where $bv_0 = bv_1 \cap c_0$. The second implication is obtained from the first one which help of (2.10), where $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow (\sum_{k=1}^{n} k a_k) \in \ell_\infty \Longrightarrow (n a_n) \in bs$ (in other words: $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longrightarrow (\Delta(\sum_{k=1}^{n} R_k^n)) \in bs \Longrightarrow (n a_n) \in bs$, since $\Delta(\sum_{k=1}^{n} R_k^n) = n a_n$ for all n). Also, the implications in (2) are immediate by those given in (1), as $cs \subset bs$ and $c \subset \ell_\infty$. For (3), it is obvious by (2.7) that $\sum_{k=1}^{n} |a_k| \leq 2 \sum_{k=1}^{n} |R_k^n| \leq 2 \sup_n \sum_{k=1}^{n} |R_k^n|$ for all n. Thus, if $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$; then $a \in \ell_1$ which proves the first implication and implies the second one. Finally, the implications in (4) are immediate from those given in (1) with $|a|$ instead of a. In other words, we have $|a_k| \leq |k a_k|$ for all k and hence $\sum_{k=1}^{\infty} |a_k| \leq \sum_{k=1}^{\infty} |k a_k|$ which yields the first implication, i.e. $(n a_n) \in \ell_1 \Longrightarrow a \in \ell_1$. The last implication can be obtained from

5

(3), where $|R_k^n| \leq \bar{R}_k^n$ $(k \leq n)$ and so $\sum_{k=1}^n |R_k^n| \leq \sum_{k=1}^n \bar{R}_k^n$, and it can also be obtained from the first implication by using (2.12), where $\left(\sum_{k=1}^n \bar{R}_k^n\right) \in \ell_\infty \Longrightarrow \left(\sum_{k=1}^n |ka_k|\right) \in \ell_\infty \Longrightarrow \left(\sum_{k=1}^n |ka_k|\right) \in c \Longrightarrow (na_n) \in \ell_1$ (note also that: $\Delta\left(\sum_{k=1}^n \bar{R}_k^n\right) = |na_n|$ for all n and so $\left(\sum_{k=1}^n \bar{R}_k^n\right) \in \ell_\infty \Longrightarrow \left(\Delta\left(\sum_{k=1}^n \bar{R}_k^n\right)\right) \in bs \Longrightarrow (|na_n|) \in bs \Longrightarrow (|na_n|) \in cs \Longrightarrow (na_n) \in \ell_1$).

Lemma 2.1 *We have the following true implications:*

(1) $(na_n) \in bs \Longrightarrow (nR_{n+1}) \in \ell_\infty$. *Also* $\left(\sum_{k=1}^n R_k^n\right) \in \ell_\infty \Longrightarrow (nR_{n+1}) \in \ell_\infty$.

(2) $(na_n) \in cs \Longrightarrow (nR_{n+1}) \in c_0$. *Also* $\left(\sum_{k=1}^n R_k^n\right) \in c \Longrightarrow (nR_{n+1}) \in c_0$.

(3) $(na_n) \in \ell_1 \Longrightarrow (n\bar{R}_{n+1}) \in c_0$. *Also* $\left(\sum_{k=1}^n \bar{R}_k^n\right) \in c \Longrightarrow (n\bar{R}_{n+1}) \in c_0$.

Proof. For (1), suppose that $(na_n) \in bs$. Then $a \in cs$ (Remark 2-2) and so the sequence $R = (R_k)$ is well-defined. Also, since $\left(\sum_{j=1}^m ja_j\right) \in \ell_\infty$ (by assumption); there is a real $M > 0$ such that $|\sum_{j=1}^m ja_j| \leq M$ for every $m \geq 1$. Thus, for all integers $m, n \geq 1$ $(m > n)$, we find that $|\sum_{j=n+1}^m ja_j| \leq 2M$ and by using (2.8) it follows that $|R_{n+1}| \leq 2M/(n+1) < 2M/n$ and hence $n|R_{n+1}| < 2M$ for all $n \geq 1$ which means that $(nR_{n+1}) \in \ell_\infty$. Also, if $\left(\sum_{k=1}^n R_k^n\right) \in \ell_\infty$, then it follows by (2.10) that $\left(\sum_{k=1}^n ka_k\right) \in \ell_\infty$ which means that $(na_n) \in bs$ and so $(nR_{n+1}) \in \ell_\infty$ (as we have already shown). For (2), assume that $(na_n) \in cs$. Then $a \in cs$ and so R exists. Also, since $\left(\sum_{j=1}^m ja_j\right) \in c$ (by hypothesis); for every positive real $\epsilon > 0$ there is an integer $k_0 > 0$ such that $|\sum_{j=n+1}^m ja_j| < \epsilon$ for all integers m and n satisfying $m > n > k_0$. This with (2.8) leads us to obtain that $|R_{n+1}| < \epsilon/(n+1) < \epsilon/n$ and hence $n|R_{n+1}| < \epsilon$ for all $n > k_0$ which means that $(nR_{n+1}) \in c_0$. Also, the other implication is obvious by (2.10). Finally, to prove (3), let $(na_n) \in \ell_1$. Then $a \in \ell_1$ (Remark 2-2) and so $\bar{R} = (\bar{R}_k)$ is well-defined. Therefore, the implications in part (3) are immediate by those of part (2) with $|a|$ instead of a, where $\bar{R}_k^n = R_k^n(|a|)$ for $n \geq k \geq 1$. This completes the proof (note that $(na_n) \in \ell_1 \Longleftrightarrow (|na_n|) \in cs$, and $|na_n| = n|a_n|$ for all n). $\square$

Remark 2-3 It must be noted that $\bar{R} \in \ell_1$ implies that $(n\bar{R}_{n+1}) \in c_0$ for every $a \in \ell_1$. To see that, we may note that $\bar{R} \in \ell_1 \Longrightarrow \left(\sum_{k=1}^n \bar{R}_k^n\right) \in \ell_\infty$ (by (4) of Remark 2-1) $\Longrightarrow \left(\sum_{k=1}^n \bar{R}_k^n\right) \in c$ (by (3) of Remark 2-1) $\Longrightarrow (n\bar{R}_{n+1}) \in c_0$ (by (3) of Lemma 2.1). In other words, it is clear that if $\bar{R} \in \ell_1$, then $\bar{R}$ is a decreasing sequence of non-negative reals such that $\bar{R} \in \ell_1$ and so $(n\bar{R}_{n+1}) \in c_0$.

Lemma 2.2 *The following conditions are equivalent to each others:*

(1) $(na_n) \in bs$.

(2) $a \in cs$, $R \in bs$ *and* $(nR_{n+1}) \in \ell_\infty$.

(3) $\left(\sum_{k=1}^n R_k^n\right) \in \ell_\infty$.

Proof. To show that given conditins are equivalent, we will prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). For this, suppose that (1) is satisfied, that is $\left(\sum_{k=1}^n ka_k\right) \in \ell_\infty$. Then $a \in cs$ (by Remark 2-2) and $(nR_{n+1}) \in \ell_\infty$ (by (1) of Lemma 2.1). Thus, it follows by (2.10) that $\left(\sum_{k=1}^n R_k\right) \in \ell_\infty$ which means that $R \in bs$ and this shows that (1) $\Longrightarrow$ (2). Also, assume that (2) is satisfied, that is $(nR_{n+1}) \in \ell_\infty$ and $R \in bs$, where $a \in cs$

6

and so $R = (R_k)$ is well-defined. Then, it follows by (2.10) that $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty$ which is (3). Lastly, it is clear by (2.10) that (3) $\Longrightarrow$ (1) (note that: (3) implies that $(\Delta(\sum_{k=1}^{n} R_k^n)) \in bs$, but $\Delta(\sum_{k=1}^{n} R_k^n) = na_n$ for all n which implies (1)). This ends the proof (note also that: each of given conditions implies that $a \in cs$). $\square$

Lemma 2.3 *The following conditions are equivalent to each others:*

(1) $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$.

(2) $a \in \ell_1$, $R \in \ell_1$ *and* $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty$.

(3) $a \in \ell_1$, $R \in \ell_1$ *and* $(nR_{n+1}) \in \ell_\infty$.

(4) $a \in \ell_1$, $R \in \ell_1$ *and* $(na_n) \in bs$.

Proof. To prove that given conditions are equivalent, it obvious by Lemma 2.2 that (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) (since $\ell_1 \subset cs \subset bs$). To see that, it is clear that if $a \in \ell_1$ and $R \in \ell_1$ (and so $a \in cs$ and $R \in bs$); then: $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty \Longleftrightarrow (nR_{n+1}) \in \ell_\infty \Longleftrightarrow (na_n) \in bs$ (see Lemma 2.2). Thus, it is remaining to prove that (1) $\Longleftrightarrow$ (2). For this, suppose that (1) is satisfied, that is $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$. Then $a \in \ell_1$ (by (3) of Remark 2-2) and it follows that $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty$ (as $\left|\sum_{k=1}^{n} R_k^n\right| \le \sum_{k=1}^{n} |R_k^n| \le \sup_n \sum_{k=1}^{n} |R_k^n|$ for all n). Thus, we obtain by Lemma 2.2 that $(nR_{n+1}) \in \ell_\infty$. Therefore, we have $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$ as well as $(nR_{n+1}) \in \ell_\infty$ and by means of (2.6), we deduce that $(\sum_{k=1}^{n} |R_k|) \in \ell_\infty$ and so $(\sum_{k=1}^{n} |R_k|) \in c$ which means that $R \in \ell_1$. Hence, we conclude that (1) $\Longrightarrow$ (2). Conversely, assume that (2) is satisfied, that is $(\sum_{k=1}^{n} R_k^n) \in \ell_\infty$ and $R \in \ell_1$, where $a \in \ell_1$ and so $a \in cs$ which means that all terms of R exist. Then, we have $(nR_{n+1}) \in \ell_\infty$ (by Lemma 2.2) and $(\sum_{k=1}^{n} |R_k|) \in \ell_\infty$ (as $R \in \ell_1$). This together with (2.6) lead us to conclude that $(\sum_{k=1}^{n} |R_k^n|) \in \ell_\infty$ which is (1), that is (2) $\Longrightarrow$ (1). Consequently, we deduce that (1) $\Longleftrightarrow$ (2) and this completes the proof (as (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4)). $\square$

Lemma 2.4 *The following conditions are equivalent to each others:*

(1) $(na_n) \in cs$.

(2) $a \in cs$, $R \in cs$ *and* $(nR_{n+1}) \in c_0$.

(3) $(\sum_{k=1}^{n} R_k^n) \in c$.

Furthermore, if any one of above conditions is satisfied, then we have

$$\sum_{k=1}^{\infty} ka_k = \lim_{n\to\infty} \sum_{k=1}^{n} R_k^n = \lim_{n\to\infty} \sum_{k=1}^{n} R_k = \sum_{k=1}^{\infty} R_k. \tag{2.13}$$

Proof. First, it is obvious, by (2) of Remark 2-2, that each one of given conditions implies that $a \in cs$ and so the sequence R is well-defined. Next, to show that these conditions are equivalent, it can easily be proved that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1), but the proof is exactly same as that of Lemma 2.2 (by using (2.10) and (2) of Lemma 2.1). Thus, we may omit the proof which is left to the reader. Further, suppose that condition (1), (2) or (3) is satisfied. Then, since these conditions are equivalent; all are satisfied and by going to the limits in all sides of (2.10) as $n \to \infty$, we get (2.13). $\square$

7

Lemma 2.5 *The following conditions are equivalent to each others:*

(1) $\left(\sum_{k=1}^{n}|R_k^n|\right) \in \ell_\infty$ *and* $(nR_{n+1}) \in c_0$.

(2) $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ *and* $(nR_{n+1}) \in c_0$.

(3) $a \in \ell_1$, $R \in \ell_1$ *and* $(nR_{n+1}) \in c_0$.

(4) $a \in \ell_1$, $R \in \ell_1$ *and* $(na_n) \in cs$.

(5) $a \in \ell_1$, $R \in \ell_1$ *and* $\left(\sum_{k=1}^{n}R_k^n\right) \in c$.

Furthermore, if any one of above conditions is satisfied, then we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}|R_k^n| = \lim_{n\to\infty}\sum_{k=1}^{n}|R_k| = \sum_{k=1}^{\infty}|R_k|. \tag{2.14}$$

Proof. To prove that given conditions are equivalent, it obvious by Lemma 2.4 that (3) $\iff$ (4) $\iff$ (5) (as $\ell_1 \subset cs$). Thus, it is remaining to prove that (1) $\iff$ (2) $\iff$ (3). For this, suppose that (1) is satisfied, that is $\left(\sum_{k=1}^{n}|R_k^n|\right) \in \ell_\infty$ and $(nR_{n+1}) \in c_0$, where $a \in \ell_1$ (by (3) of Remark 2-2) and so R is well-defined. Also, since $\left(\sum_{k=1}^{n}|R_k^n|\right) \in \ell_\infty$; we get $R \in \ell_1$ (by Lemma 2.3) which means that $\left(\sum_{k=1}^{n}|R_k|\right) \in c$. Therefore, we have $(nR_{n+1}) \in c_0$ as well as $\left(\sum_{k=1}^{n}|R_k|\right) \in c$. This leads us with help of (2.6) to deduce that $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ which means that (2) is satisfied and hence (1) $\Longrightarrow$ (2). Also, it is trivial that (2) $\Longrightarrow$ (1) and so (1) $\iff$ (2). Next, assume that (2) is satisfied, that is $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ and $(nR_{n+1}) \in c_0$, where $a \in \ell_1$ (by (3) of Remark 2-2) and so R is well-defined. Then, by using (2.6), it can easily be seen that $\left(\sum_{k=1}^{n}|R_k|\right) \in c$ and so $R \in \ell_1$, that is (2) $\Longrightarrow$ (3). Conversely, suppose that (3) holds, that is $a \in \ell_1$, $R \in \ell_1$ and $(nR_{n+1}) \in c_0$. Then, it follows by (2.6) that $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ which means that (2) holds, that is (3) $\Longrightarrow$ (2) and hence (2) $\iff$ (3). Consequently, the given conditions are all equivalent to each others. Finally, if any one of these equivalent conditions is satisfied; we have $(nR_{n+1}) \in c_0$, $\left(\sum_{k=1}^{n}|R_k^n|\right) \in c$ and $\left(\sum_{k=1}^{n}|R_k|\right) \in c$. Therefore, from (2.6) we get (2.14) and this completes the proof. □

Finally, we end this section with the following last lemma for which we need to keep in mind those facts mentioned in Remark 2-1 with our notations given by (2.11).

Lemma 2.6 *The following conditions are equivalent to each others:*

(1) $\left(\sum_{k=1}^{n}\bar{R}_k^n\right) \in \ell_\infty$.

(2) $\left(\sum_{k=1}^{n}\bar{R}_k^n\right) \in c$.

(3) $(na_n) \in \ell_1$.

(4) $a \in \ell_1$ *and* $\bar{R} \in \ell_1$.

Furthermore, if any one of above conditions is satisfied, then $(n\bar{R}_{n+1}) \in c_0$ *and we have*

$$\sum_{k=1}^{\infty}|ka_k| = \lim_{n\to\infty}\sum_{k=1}^{n}\bar{R}_k^n = \lim_{n\to\infty}\sum_{k=1}^{n}\bar{R}_k = \sum_{k=1}^{\infty}\bar{R}_k. \tag{2.15}$$

8

Proof. First, it is obvious that each of given conditions implies that $a \in \ell_1$ (by (4) of Remark 2-2). Next, to show that these conditins are equivalent, we will prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1). For this, suppose that (1) is satisfied, that is $(\sum_{k=1}^n \bar{R}_k^n) \in \ell_\infty$. Then, it follows by (3) of Remark 2-1 that the sequence $(\sum_{k=1}^n \bar{R}_k^n)$ is increasing as well as bounded. This implies that $(\sum_{k=1}^n \bar{R}_k^n) \in c$ which is (2). Hence (1) $\Longrightarrow$ (2). Also, let (2) be satisfied, that is $(\sum_{k=1}^n \bar{R}_k^n) \in c$. Then, it follows by (2.12) that $(\sum_{k=1}^n |ka_k|) \in c$ and so $(na_n) \in \ell_1$ which is (3), that is (2) $\Longrightarrow$ (3). Further, assume that (3) is satisfied, that is $(na_n) \in \ell_1$. Then $a \in \ell_1$ (by (4) of Remark 2-2) and so $\bar{R}$ is well-defined. Further, we have, by assumption, that $(|na_n|) \in cs$ or $(n|a_n|) \in cs$. Thus, it follows, by Lemma 2.4 with $|a|$ instead of a, that $R(|a|) \in cs$ and so $\bar{R} \in cs$, where $\bar{R} = R(|a|)$ by (2.11). This together with (2) of Remark 2-1 implies that $\bar{R} \in \ell_1$ which means that (4) is satisfied, that is (3) $\Longrightarrow$ (4). Moreover, it is obvious, by (4) of Remark 2-1, that (4) $\Longrightarrow$ (1). Therefore, all given conditions are equivalent to each others. Finally, if any one of these equivalent conditions is satisfied; we then have $(\sum_{k=1}^n \bar{R}_k^n) \in c$, $(\sum_{k=1}^n \bar{R}_k) \in c$ and $(n\bar{R}_{n+1}) \in c_0$ (see (3) of Lemma 2.1 and Remark 2-3). Therefore, by going to the limits in all sides of (2.12) as $n \to \infty$, we get (2.15) and this ends the proof. □

Remark 2-4 It must be noted that every condition in Lemma 2.3, 2.5 or 2.6 implies that $a \in \ell_1$ which means that $a \in \ell_1$ is necessary for each of those conditions to be satisfied. To see that, we have $\sum_{k=1}^\infty |a_k| = \sum_{k=1}^\infty |\Delta(R_{k+1})|$, that is $R \in \ell_1 \Longrightarrow a \in \ell_1$ (as $\ell_1 \subset bv_1$). Also, since $|a_k| \leq \bar{R}_k$ for all $k \geq 1$; we get $\sum_{k=1}^\infty |a_k| \leq \sum_{k=1}^\infty \bar{R}_k$, that is $\bar{R} \in \ell_1 \Longrightarrow a \in \ell_1$ (see also (3) and (4) of Remark 2-2). In other words, every condition in Lemma 2.3, 2.5 or 2.6 cannot be held if $a \notin \ell_1$. Similarly, we must note that every condition in Lemma 2.2 or 2.4 implies the necessary condition $a \in cs$.

3 The α-, β- and γ-Dualities

In the present section, we apply the results of previous section to conclude the α-, β- and γ-duals of the difference spaces $\mu(\Delta)$ and λ-difference spaces $\mu(\Delta^\lambda)$, where μ stands for any one of the spaces c_0, c or ℓ_∞.

By θ, we mean any one of the duality symbols α, β or γ, that is $\theta := \alpha$, β or γ. Also, by θ-dual of a sequence space X, we mean the α-, β- or γ-dual of X, that is $X^\theta = \{a \in w : ax \in \langle\theta\rangle \text{ for all } x \in X\}$, where $\langle\alpha\rangle = \ell_1$, $\langle\beta\rangle = cs$ and $\langle\gamma\rangle = bs$. For example, it is known that $\mu^\theta = \ell_1$, and since $\mu \subset \mu(\Delta)$ and $\mu \subset \mu(\Delta^\lambda)$; we have $\{\mu(\Delta)\}^\theta \subset \mu^\theta$ and $\{\mu(\Delta^\lambda)\}^\theta \subset \mu^\theta$ which leads us to deduce the following inclusions:

$$\{\mu(\Delta)\}^\theta \subset \ell_1 \quad \text{and} \quad \{\mu(\Delta^\lambda)\}^\theta \subset \ell_1.$$

Thus, we assume that $a \in \ell_1$ and we may begin with obtaining θ-duals of the difference spaces $\mu(\Delta)$ which will be needed to deduce θ-duals of the λ-difference spaces $\mu(\Delta^\lambda)$. For this, we will use our notations given by (2.5) and (2.11). Also, for every $x \in w$, it is known that $x \in \mu(\Delta) \Longleftrightarrow \Delta(x) \in \mu$, and by using (2.9) we can write the following:

$$\sum_{k=1}^n a_k x_k = \sum_{k=1}^n R_k^n \Delta(x_k) = A_n(\Delta(x)) \qquad (n \geq 1),$$

where $A = [a_{nk}]$ is a triangle defined for all $n, k \geq 1$ by $a_{nk} = R_k^n$ when $1 \leq k \leq n$ and

9

$a_{nk} = 0$ when $k > n \geq 1$. Therefore, by using above relation, we deduce the following:

$$\begin{aligned} a \in \{\mu(\Delta)\}^{\gamma} &\iff ax \in bs \text{ for all } x \in \mu(\Delta) \\ &\iff \left(\sum_{k=1}^{n} a_k x_k\right) \in \ell_{\infty} \text{ for all } x \in \mu(\Delta) \\ &\iff A(\Delta(x)) \in \ell_{\infty} \text{ for all } \Delta(x) \in \mu \\ &\iff A(y) \in \ell_{\infty} \text{ for all } y \in \mu \quad (\, y = \Delta(x)\,) \\ &\iff A \in (\mu, \ell_{\infty}) \quad (A \text{ is a triangle} \implies A(y) \text{ always exists}). \end{aligned}$$

On other side, it is well-known that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ is the necessary and sufficient condition in order that $A \in (\mu, \ell_{\infty})$ (see [15]). Thus, it follows, by definition of our triangle A, that $A \in (\mu, \ell_{\infty}) \iff \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \iff \sup_n \sum_{k=1}^{n} |R_k^n| < \infty$. Therefore, we deduce that

$$a \in \{\mu(\Delta)\}^{\gamma} \iff \left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_{\infty}. \tag{3.1}$$

Similarly, it can easily be shown that

$$a \in \{\mu(\Delta)\}^{\alpha} \iff \left(\sum_{k=1}^{n} \bar{R}_k^n\right) \in \ell_{\infty} \tag{3.2}$$

$$a \in \{c_0(\Delta)\}^{\beta} \iff \left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_{\infty} \tag{3.3}$$

$$a \in \{\eta(\Delta)\}^{\beta} \iff \left(\sum_{k=1}^{n} |R_k^n|\right) \in \ell_{\infty} \text{ and } (nR_{n+1}) \in c_0, \tag{3.4}$$

where η stands for any one of the spaces c or ℓ_{∞}, and so $\eta(\Delta)$ is the respective one of the spaces $c(\Delta)$ or $\ell_{\infty}(\Delta)$. Consequently, with help of Lemmas 2.3, 2.5 and 2.6, we conclude the following result which gives two formulae for each type of duals of $\mu(\Delta)$.

Theorem 3.1 *Let $\mu(\Delta)$ be any one of the spaces $c_0(\Delta)$, $c(\Delta)$ or $\ell_{\infty}(\Delta)$. Then, we have the following:*

(1) *The α-duals of $\mu(\Delta)$ are given by*

$$\{\mu(\Delta)\}^{\alpha} = \{a \in \ell_1 : \bar{R} \in \ell_1\} = \{a \in \ell_1 : (na_n) \in \ell_1\}.$$

(2) *The β-duals of $\mu(\Delta)$ are given by*

$$\begin{aligned} \{c_0(\Delta)\}^{\beta} &= \{a \in \ell_1 : (nR_{n+1}) \in \ell_{\infty} \text{ and } R \in \ell_1\} \\ &= \{a \in \ell_1 : (na_n) \in bs \text{ and } R \in \ell_1\}. \end{aligned}$$

$$\begin{aligned} \{\eta(\Delta)\}^{\beta} &= \{a \in \ell_1 : (nR_{n+1}) \in c_0 \text{ and } R \in \ell_1\} \\ &= \{a \in \ell_1 : (na_n) \in cs \text{ and } R \in \ell_1\}, \end{aligned}$$

where $\eta(\Delta)$ stands for any of the spaces $c(\Delta)$ or $\ell_{\infty}(\Delta)$.

(3) *The γ-duals of $\mu(\Delta)$ are given by*

$$\begin{aligned} \{\mu(\Delta)\}^{\gamma} &= \{a \in \ell_1 : (nR_{n+1}) \in \ell_{\infty} \text{ and } R \in \ell_1\} \\ &= \{a \in \ell_1 : (na_n) \in bs \text{ and } R \in \ell_1\}, \end{aligned}$$

where $\bar{R} = (\bar{R}_n)$ and $R = (R_n)$ such that $\bar{R}_n = \sum_{j=n}^{\infty} |a_j|$ and $R_n = \sum_{j=n}^{\infty} a_j$ for all n.

10

Proof. Part (1) is immediate by combining (3.2) with Lemma 2.6 (see [5, Theorem 2.1] for the 2nd formula of $\{\mu(\Delta)\}^\alpha$). For part (2), the first two formulae of $\{c_0(\Delta)\}^\beta$ are obtained from (3.3) with help of Lemma 2.3 (see [6, Lemma 3] for the 1st formula of $\{c_0(\Delta)\}^\beta$). Also, the second two formulae of $\{\eta(\Delta)\}^\beta$ are obtained from (3.4) and Lemma 2.5 (see [5, Theorem 2.1] for the 2nd formula of $\{\eta(\Delta)\}^\beta$). Lastly, part (3) is immediate by (3.1) with Lemma 2.3 (see [5, Theorem 2.1] for the 2nd formula). □

Remark 3-1 In (1) of Theorem 3.1, according to Lemma 2.6, we can write "$a \in w$" instead of "$a \in \ell_1$" in only the second formula of α-dual of $\mu(\Delta)$. That is, we may write $\{\mu(\Delta)\}^\alpha = \{a \in w : (na_n) \in \ell_1\}$, but it is understood that $a \in \ell_1$ which was our assumption from the beginning and so there is no contradiction with using "$a \in \ell_1$" for similarity with other formulae in which the term "$a \in \ell_1$" is necessary and must be mentioned (see Remark 2-4). For example, in the first formula of α-dual of $\mu(\Delta)$, the sequence $\bar{R} = (\bar{R}_k)$ cannot be defined if $a \notin \ell_1$, where $\bar{R}_k = \sum_{j=k}^{\infty}|a_j|$ for all k. On other side, we may note that $\{c_0(\Delta)\}^\beta = \{c_0(\Delta)\}^\gamma$, $\{c(\Delta)\}^\theta = \{\ell_\infty(\Delta)\}^\theta$ for $\theta = \alpha$, β and γ, while $\{c_0(\Delta)\}^\theta = \{\eta(\Delta)\}^\theta$ for only $\theta = \alpha$ and γ (not β), where $\eta = c$ or ℓ_∞.

Corollary 3.2 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta)\}^\alpha$; then $(n\bar{R}_{n+1}) \in c_0$, $(\bar{R}_{n+1}\sigma_n(y)) \in c_0$ and $(\bar{R}_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$.*

(2) *If $a \in \{\mu(\Delta)\}^\beta$; then $(R_{n+1}\sigma_n(y)) \in c_0$ and $(R_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$.*

(3) *If $a \in \{\mu(\Delta)\}^\gamma$; then $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ and $(R_{n+1}\sigma_n(|y|)) \in \ell_\infty$ for all $y \in \mu$.*

Proof. For (1), let $a \in \{\mu(\Delta)\}^\alpha$. Then $a \in \ell_1$ and $\bar{R} \in \ell_1$ (by Theorem 3.1) and so $(n\bar{R}_{n+1}) \in c_0$ (by Remark 2-3 or Lemma 2.6). Thus, for every $y \in \mu \subset \ell_\infty$, we have $|y_n| \le \|y\|_\infty < \infty$ and so $|\sigma_n(y)| \le \sigma_n(|y|) \le n\|y\|_\infty$ for all n. This implies that $0 \le \bar{R}_{n+1}|\sigma_n(y)| \le \bar{R}_{n+1}\sigma_n(|y|) \le n\bar{R}_{n+1}\|y\|_\infty \to 0$ as $n \to \infty$ which proves (1). For (2), let $a \in \{\mu(\Delta)\}^\beta$ and take any $y \in \mu$ which implies that $(\sigma_n(y)/n) \in \mu$. Then, one of the sequences (nR_{n+1}) or $(\sigma_n(y)/n)$ is bounded and the other tends to zero. Thus $(R_{n+1}\sigma_n(y)) = (nR_{n+1})(\sigma_n(y)/n) \in c_0$, that is $(R_{n+1}\sigma_n(y)) \in c_0$ for all $y \in \mu$ and hence $(R_{n+1}\sigma_n(|y|)) \in c_0$ for all $y \in \mu$ (as $y \in \mu \Longrightarrow |y| \in \mu$). Finally, part (3) can be proved same as part (1) with $(nR_{n+1}) \in \ell_\infty$ instead of $(n\bar{R}_{n+1}) \in c_0$. □

Remark 3-2 As in the proof of Corollary 3.2, it can easily be shown the following:

$$\begin{aligned}\{\mu(\Delta)\}^\beta &= \{a \in \ell_1 : R \in \ell_1 \text{ and } (x_nR_{n+1}) \in c_0 \text{ for all } x \in \mu(\Delta)\}\\ &= \{a \in \ell_1 : R \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in c_0 \text{ for all } y \in \mu\},\end{aligned}$$

$$\begin{aligned}\{\mu(\Delta)\}^\gamma &= \{a \in \ell_1 : R \in \ell_1 \text{ and } (x_nR_{n+1}) \in \ell_\infty \text{ for all } x \in \mu(\Delta)\}\\ &= \{a \in \ell_1 : R \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in \ell_\infty \text{ for all } y \in \mu\},\end{aligned}$$

with noting that all these formulae are equal to each others when $\mu = c_0$.

Now, we turn to the θ-duals of the spaces $\mu(\Delta^\lambda)$. As usual, our notations given by (2.5) and (2.11) will be used, where $a \in \ell_1$ (as $\{\mu(\Delta^\lambda)\}^\theta \subset \ell_1$). Besides, we define the sequence $v = (v_k)$ of non-negative reals as follows:

$$v_k = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} = \frac{\lambda_{k-1}}{\Delta(\lambda_k)} \qquad (k \ge 1). \tag{3.5}$$

11

Further, for any sequence $x = (x_k) \in w$, let $y = (y_k)$ be the sequence connected with x by the relation $y = \tilde{\Lambda}(x)$ which means that

$$y_k = \tilde{\Lambda}_k(x) = \Lambda_k(x) - \Lambda_{k-1}(x) \qquad (k \geq 1).$$

Then, with help of (2.2), it can easily be seen that $x_k = \Delta(\lambda_k \sum_{j=1}^{k} y_j)/\Delta(\lambda_k)$ $(k \geq 1)$ which can equivalently be written as follows:

$$x_k = v_k y_k + \sum_{j=1}^{k} y_j \qquad (k \geq 1). \tag{3.6}$$

It is obvious that x and y are connected by $y = \tilde{\Lambda}(x)$ if and only if (3.6) is satisfied. Also $x \in \mu(\Delta^\lambda)$ if and only if $y \in \mu$, and we have $\|x\|_{\Delta^\lambda} = \|y\|_\infty$. In fact, for every $x \in \mu(\Delta^\lambda)$ there exists a unique $y \in \mu$ given by $y = \tilde{\Lambda}(x)$ and conversely, for every $y \in \mu$ there exists a unique $x \in \mu(\Delta^\lambda)$ given by (3.6). Thus, here and in what follows, we assume that x and y are connected by $y = \tilde{\Lambda}(x)$ which implies the validity of (3.6) by which we find that $a_k x_k = a_k v_k y_k + a_k \sum_{j=1}^{k} y_j$ $(k \geq 1)$ and so we obtain that

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (a_k v_k + R_k^n) y_k \quad \text{and} \quad \sum_{k=1}^{n} |a_k x_k| \leq \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k^n)|y_k|$$

which can be used to derive the following relations in which $n \geq 1$:

$$\sum_{k=1}^{n} |a_k x_k| \leq \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k)|y_k| - \bar{R}_{n+1}\sigma_n(|y|), \tag{3.7}$$

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} (a_k v_k + R_k) y_k - R_{n+1}\sigma_n(y), \tag{3.8}$$

$$\left|\sum_{k=1}^{n} a_k x_k\right| \leq \sum_{k=1}^{n} |(a_k v_k + R_k) y_k| + |R_{n+1}\sigma_n(y)|. \tag{3.9}$$

Furthermore, let $x = (x_k)$ be given by $x_k = k + v_k = \Delta(k\lambda_k)/\Delta(\lambda_k)$ for all k. Then, it follows by (2.4) that $y_k = \tilde{\Lambda}_k(x) = 1$ $(k \geq 1)$. Thus, by taking $x_k = k + v_k$ and $y_k = 1$ in (3.8) and (3.9) for all k, we get the following:

$$\sum_{k=1}^{n} (k + v_k) a_k = \sum_{k=1}^{n} (a_k v_k + R_k^n) = \sum_{k=1}^{n} (a_k v_k + R_k) - nR_{n+1}, \tag{3.10}$$

$$\left|\sum_{k=1}^{n} (k + v_k) a_k\right| = \left|\sum_{k=1}^{n} (a_k v_k + R_k^n)\right| \leq \sum_{k=1}^{n} |a_k v_k + R_k| + |nR_{n+1}| \tag{3.11}$$

and on replacing a by $|a|$ in (3.10), we obtain that

$$\sum_{k=1}^{n} (k + v_k)|a_k| = \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k^n) = \sum_{k=1}^{n} (|a_k v_k| + \bar{R}_k) - n\bar{R}_{n+1}. \tag{3.12}$$

On other side, it is obvious that $c_0(\Delta^\lambda) \subset \mu(\Delta^\lambda)$ and we have $c_0(\Delta) \subset c_0(\Delta^\lambda)$ which yields that $c_0(\Delta) \subset \mu(\Delta^\lambda)$ (see [14, Theorem 3.10]). Therefore, we deduce the following satisfied inclusions:

$$\{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta \subset \{c_0(\Delta)\}^\gamma \quad \text{and} \quad \{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta^\lambda)\}^\gamma. \tag{3.13}$$

Moreover, we now prove the following result which shows that $a \in \{\mu(\Delta)\}^\theta$ and $av \in \ell_1$ are the necessary conditions in order that $a \in \{\mu(\Delta^\lambda)\}^\theta$, where $\theta = \alpha$, β or γ, and $\mu = c_0$, c or ℓ_∞ (it will be shown latter that these conditions are also sufficient).

12

Lemma 3.3 *We have the following facts:*

(1) *If $a \in \{\mu(\Delta^\lambda)\}^\theta$; then $av = (a_k v_k) \in \ell_1$.*

(2) *The inclusion $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ always holds.*

Proof. For (1), let $a \in \{\mu(\Delta^\lambda)\}^\theta$ be arbitrary. Then, it follows by (3.13) that $a \in \{c_0(\Delta)\}^\gamma$ as well as $a \in \{c_0(\Delta^\lambda)\}^\gamma$. Thus $a \in \ell_1$ and $R \in \ell_1$ (as $a \in \{c_0(\Delta)\}^\gamma$ by Theorem 3.1). Also, for any $y \in c_0$, let $x = (x_k)$ be given by (3.6). Then $x \in c_0(\Delta^\lambda)$ and since $a \in \{c_0(\Delta^\lambda)\}^\gamma$; we get $ax \in bs$ and so $(\sum_{k=1}^n a_k x_k) \in \ell_\infty$. Further, it follows by (3) of Corollary 3.2 that $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ (since $y \in c_0$ and $a \in \{c_0(\Delta)\}^\gamma$). Therefore, we have shown that $(\sum_{k=1}^n a_k x_k) \in \ell_\infty$ as well as $(R_{n+1}\sigma_n(y)) \in \ell_\infty$ which together with (3.8) imply that $(\sum_{k=1}^n (a_k v_k + R_k) y_k) \in \ell_\infty$ and this means that $(av + R)y \in bs$ for all $y \in c_0$ (as $y \in c_0$ was arbitrary). Hence, we deduce that $av + R \in c_0^\gamma = \ell_1$ and so $av \in \ell_1$ (as $R \in \ell_1$). Next, to prove (2), it is obvious that given inclusion is trivially satisfied when $\theta = \alpha$ or γ. To see that, we have $\{c_0(\Delta)\}^\theta = \{\mu(\Delta)\}^\theta$ when $\theta = \alpha$ or γ (by Theorem 3.1), but $\{\mu(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta$ by (3.13) and so $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ for $\theta = \alpha$ or γ. On other side, consider the case $\theta = \beta$. Then, it is clear by (3.13) that $\{c_0(\Delta^\lambda)\}^\beta \subset \{c_0(\Delta)\}^\beta$. Thus, it is remaining to show that $\{\eta(\Delta^\lambda)\}^\beta \subset \{\eta(\Delta)\}^\beta$, where $\eta = c$ or ℓ_∞. For this, take any $a \in \{\eta(\Delta^\lambda)\}^\beta$. Then $av \in \ell_1$ (by part (1)) and from (3.13) we find that $a \in \{c_0(\Delta)\}^\beta$ and so $a \in \ell_1$ as well as $R \in \ell_1$ by Theorem 3.1. Besides, we have $\tilde{\Lambda}(k + v_k) = (1, 1, 1, \cdots) \in \eta$ which means that $(k + v_k) \in \eta(\Delta^\lambda)$ and since $a \in \{\eta(\Delta^\lambda)\}^\beta$; we obtain that $(k + v_k)a \in cs$, that is $(ka_k + a_k v_k) \in cs$ and hence $(ka_k) \in cs$ (as $av \in \ell_1 \subset cs$). Therefore, we have already shown that $a \in \ell_1$, $(ka_k) \in cs$ and $R \in \ell_1$ which leads us with Theorem 3.1 to deduce that $a \in \{\eta(\Delta)\}^\beta$. This means that $\{\eta(\Delta^\lambda)\}^\beta \subset \{\eta(\Delta)\}^\beta$ which completes the proof. □

Theorem 3.4 *For $\theta = \alpha$, β or γ, the θ-duals of the spaces $\mu(\Delta^\lambda)$ are given by*

$$\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta \cap \{a \in w : av \in \ell_1\},$$

where $v = (\lambda_{k-1}/\Delta(\lambda_k))$ and μ stands for any one of the spaces c_0, c or ℓ_∞.

Proof. We will prove that $\{\mu(\Delta^\lambda)\}^\theta = D^\theta$, where $D^\theta = \{\mu(\Delta)\}^\theta \cap \{a \in w : av \in \ell_1\}$. For this, it is obvious by Lemma 3.3 that $\{\mu(\Delta^\lambda)\}^\theta \subset D^\theta$. Thus, we have to prove the converse inclusion. So, let $a \in D^\theta$ be arbitrary and let's show that $a \in \{\mu(\Delta^\lambda)\}^\theta$. For, take any $x \in \mu(\Delta^\lambda)$ and let $y = (y_k)$ be the sequence connected by $y = \tilde{\Lambda}(x)$. Then $y \in \mu$ and since $a \in D^\theta$; we have $av \in \ell_1$ and $a \in \{\mu(\Delta)\}^\theta$. Therefore, we have three distinct cases which are $\theta = \alpha$, $\theta = \beta$ and $\theta = \gamma$. In the first case ($\theta = \alpha$), we have $a \in \{\mu(\Delta)\}^\alpha$ and so $\bar{R} \in \ell_1$ (by Theorem 3.1), where $a \in \ell_1$. In such case, we have $y \in \mu$, $av \in \ell_1$ and $\bar{R} \in \ell_1$. Thus, we deduce that $(\sum_{k=1}^n (|a_k v_k| + \bar{R}_k)|y_k|) \in c$ and $(\bar{R}_{n+1}\sigma_n(|y|)) \in c_0$ by (1) of Corollary 3.2. Therefore, it follows by (3.7) that $(\sum_{k=1}^n |a_k x_k|) \in \ell_\infty$ and so $(\sum_{k=1}^n |a_k x_k|) \in c$ which means that $ax \in \ell_1$ for all $x \in \mu(\Delta^\lambda)$ and hence $a \in \{\mu(\Delta^\lambda)\}^\alpha$ which implies that $D^\alpha \subset \{\mu(\Delta^\lambda)\}^\alpha$. Similarly, in the second case ($\theta = \beta$), we have $a \in \{\mu(\Delta)\}^\beta$ and so $R \in \ell_1$ (by Theorem 3.1) as well as $(R_{n+1}\sigma_n(y)) \in c_0$ by (2) of Corollary 3.2, where $a \in \ell_1$. Also, since $y \in \mu \subset \ell_\infty$, $av \in \ell_1$ and $R \in \ell_1$, we deduce that $(\sum_{k=1}^n (a_k v_k + R_k) y_k) \in c$ and it follows by (3.8) that $(\sum_{k=1}^n a_k x_k) \in c$ which means that $ax \in cs$ for all $x \in \mu(\Delta^\lambda)$ and so $a \in \{\mu(\Delta^\lambda)\}^\beta$ which implies that $D^\beta \subset \{\mu(\Delta^\lambda)\}^\beta$.

13

Finally, in the third case ($\theta = \gamma$), we have $y \in \mu \subset \ell_\infty$, $av \in \ell_1$ and $R \in \ell_1$ $(a \in \ell_1)$. Thus, we deduce that $(\sum_{k=1}^{n} |(a_k v_k + R_k) y_k|) \in \ell_\infty$ and $(R_{n+1}\sigma_n(y)) \in \ell_\infty$. Hence, it follows by (3.9) that $(\sum_{k=1}^{n} a_k x_k) \in \ell_\infty$ which means that $ax \in bs$ for all $x \in \mu(\Delta^\lambda)$ and so $a \in \{\mu(\Delta^\lambda)\}^\gamma$ which implies that $D^\gamma \subset \{\mu(\Delta^\lambda)\}^\gamma$. Consequently $D^\theta \subset \{\mu(\Delta^\lambda)\}^\theta$ which yields that $\{\mu(\Delta^\lambda)\}^\theta = D^\theta$ and we have done. □

Furthermore, let $\rho = \beta$ or γ. Then, it follows by Theorem 3.4 that

$$a \in \{\mu(\Delta^\lambda)\}^\rho \Longleftrightarrow a \in \{\mu(\Delta)\}^\rho \text{ and } av \in \ell_1$$

which can equivalently be written as follows:

$$a \in \{\mu(\Delta^\lambda)\}^\rho \Longleftrightarrow a \in \{\mu(\Delta)\}^\rho \text{ and } av + R \in \ell_1.$$

To see that, it is obvious that if $R \in \ell_1$; then: $av \in \ell_1 \Longleftrightarrow av + R \in \ell_1$. That is $av \in \ell_1 \Longleftrightarrow av + R \in \ell_1$ (provided that $R \in \ell_1$). Besides, we have $R \in \ell_1$ in both sides of above equivalence, since each of $a \in \{\mu(\Delta)\}^\rho$ or $a \in \{\mu(\Delta^\lambda)\}^\rho$ implies $R \in \ell_1$ by Theorem 3.1 and Lemma 3.3. This leads us to conclude the following:

Remark 3-3 For $\rho = \beta$ or γ, the ρ-duals of the spaces $\mu(\Delta^\lambda)$ are given by

$$\{\mu(\Delta^\lambda)\}^\rho = \{\mu(\Delta)\}^\rho \cap \{a \in w : av + R(a) \in \ell_1\},$$

where $R(a)$ and v are respectively given by (2.5) and (3.5).

Theorem 3.5 *Let $\mu(\Delta^\lambda)$ be any one of the spaces $c_0(\Delta^\lambda)$, $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$. Then, we have the following:*

(1) *The α-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\alpha &= \{a \in \ell_1 : \bar{R} \in \ell_1 \text{ and } av \in \ell_1\}\\ &= \{a \in \ell_1 : (ka_k) \in \ell_1 \text{ and } av \in \ell_1\}.\end{aligned}$$

(2) *The β-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{c_0(\Delta^\lambda)\}^\beta &= \{a \in \ell_1 : (kR_{k+1}) \in \ell_\infty, R \in \ell_1 \text{ and } av \in \ell_1\}\\ &= \{a \in \ell_1 : (ka_k) \in bs, R \in \ell_1 \text{ and } av \in \ell_1\}.\end{aligned}$$

$$\begin{aligned}\{\eta(\Delta^\lambda)\}^\beta &= \{a \in \ell_1 : (kR_{k+1}) \in c_0, R \in \ell_1 \text{ and } av \in \ell_1\}\\ &= \{a \in \ell_1 : (ka_k) \in cs, R \in \ell_1 \text{ and } av \in \ell_1\},\end{aligned}$$

where $\eta(\Delta^\lambda)$ stands for any of the spaces $c(\Delta^\lambda)$ or $\ell_\infty(\Delta^\lambda)$.

(3) *The γ-duals of $\mu(\Delta^\lambda)$ are given by*

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\gamma &= \{a \in \ell_1 : (kR_{k+1}) \in \ell_\infty, R \in \ell_1 \text{ and } av \in \ell_1\}\\ &= \{a \in \ell_1 : (ka_k) \in bs, R \in \ell_1 \text{ and } av \in \ell_1\},\end{aligned}$$

where $v = (\lambda_{k-1}/\Delta(\lambda_k))$, $\bar{R} = (\bar{R}_k)$ and $R = (R_k)$ such that $\bar{R}_k = \sum_{j=k}^{\infty} |a_j|$ and $R_k = \sum_{j=k}^{\infty} a_j$ for all $n \geq 1$.

14

Proof. It is immediate by combining Theorems 3.1 and 3.4 . □

Remark 3-4 In Theorem 3.5, except for the second formula of the α-dual of $\mu(\Delta^\lambda)$, the term "$a \in \ell_1$" is not redundant or superfluous, as we have seen in Remarks 2-4 and 3-1). Besides, we note that $\{c_0(\Delta^\lambda)\}^\beta = \{c_0(\Delta^\lambda)\}^\gamma$, $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta$ for $\theta = \alpha$, β and γ, while $\{c_0(\Delta^\lambda)\}^\theta = \{\eta(\Delta^\lambda)\}^\theta$ for only $\theta = \alpha$ and γ (not β), where $\eta = c$ or ℓ_∞.

Moreover, let x and y be connected by $y = \tilde{\Lambda}(x)$. Then $x \in \mu(\Delta^\lambda) \iff y \in \mu$, and by using (3.6) we obtain that

$$\frac{x_k}{k + v_k} = y_k - \frac{k}{k+v_k}\Big(y_k - \frac{1}{k}\,\sigma_k(y)\Big) \qquad (k \geq 1).$$

Thus, we deduce that: (If $x \in \mu(\Delta^\lambda)$; then $(\, x_k/(k+v_k)\,) \in \mu \subset \ell_\infty$. In particular, if $x \in c(\Delta^\lambda)$; then $\lim_{k\to\infty} \tilde{\Lambda}_k(x) = \lim_{k\to\infty} x_k/(k+v_k)$) which is analogous to the fact: (If $x \in \mu(\Delta)$; then $(\, x_k/k\,) \in \mu \subset \ell_\infty$. In particular, if $x \in c(\Delta)$; then $\lim_{k\to\infty} \Delta(x_k) = \lim_{k\to\infty} x_k/k$). Hence, by using these facts and Remark 3-2, we deduce the following:

Remark 3-5 We have the following:

$$\{\mu(\Delta^\lambda)\}^\alpha = \{a \in \ell_1 : (\,(k+v_k)a_k\,) \in \ell_1\} = \{(\, a_k/(k+v_k)\,) : a = (a_k) \in \ell_1\},$$

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\beta &= \{a \in \ell_1 : R \in \ell_1,\, av \in \ell_1 \text{ and } (x_n R_{n+1}) \in c_0 \text{ for all } x \in \mu(\Delta)\}\\ &= \{a \in \ell_1 : R \in \ell_1,\, av \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in c_0 \text{ for all } y \in \mu\},\end{aligned}$$

$$\begin{aligned}\{\mu(\Delta^\lambda)\}^\gamma &= \{a \in \ell_1 : R \in \ell_1,\, av \in \ell_1 \text{ and } (x_n R_{n+1}) \in \ell_\infty \text{ for all } x \in \mu(\Delta)\}\\ &= \{a \in \ell_1 : R \in \ell_1,\, av \in \ell_1 \text{ and } (R_{n+1}\sigma_n(|y|)) \in \ell_\infty \text{ for all } y \in \mu\}.\end{aligned}$$

Corollary 3.6 *We have the following facts:*
(1) *If $a \in \{\mu(\Delta^\lambda)\}^\alpha$; then $(n\bar{R}_{n+1}) \in c_0$ and we have the following:*

$$\sum_{k=1}^{\infty}(k+v_k)|a_k| = \lim_{n\to\infty}\sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k^n) = \sum_{k=1}^{\infty}(|a_k v_k| + \bar{R}_k),$$

$$\sum_{k=1}^{\infty}|a_k x_k| \leq \lim_{n\to\infty}\sum_{k=1}^{n}(|a_k v_k| + \bar{R}_k^n)|y_k| = \sum_{k=1}^{\infty}(|a_k v_k| + \bar{R}_k)|y_k|,$$

where $x \in \mu(\Delta^\lambda)$ and $y = \tilde{\Lambda}(x)$.
(2) *If $a \in \{\mu(\Delta^\lambda)\}^\gamma$; then we have the following:*

$$\sup_n \sum_{k=1}^{n}|a_k v_k + R_k^n| \leq \sum_{k=1}^{\infty}|a_k v_k + R_k| + \sup_n |nR_{n+1}| < \infty,$$

$$\sup_n \Big|\sum_{k=1}^{n}(k+v_k)a_k\Big| \leq \sum_{k=1}^{\infty}|a_k v_k + R_k| + \sup_n |nR_{n+1}| < \infty,$$

$$\sup_n \Big|\sum_{k=1}^{n}a_k x_k\Big| \leq \sum_{k=1}^{\infty}|(a_k v_k + R_k)y_k| + \sup_n |R_{n+1}\sigma_n(y)| < \infty,$$

where $x \in \mu(\Delta^\lambda)$ and $y = \tilde{\Lambda}(x)$.

Proof. We have $\{\mu(\Delta^\lambda)\}^\theta \subset \{\mu(\Delta)\}^\theta$ by Lemma 3.3. Thus, by using Corollary 3.2 and Remark 3-5, we deduce this result, where part (1) is immediate by (3.7) and (3.12), and part (2) is obtained from (3.9) and (3.11). □

15

Corollary 3.7 *We have the following facts:*
(1) *If $a \in \{\mu(\Delta^\lambda)\}^\beta$; then for every $x \in \mu(\Delta^\lambda)$ with $y = \tilde{\Lambda}(x)$, we have*

$$\sum_{k=1}^{\infty} a_k x_k = \lim_{n\to\infty} \sum_{k=1}^{n} (a_k v_k + R_k^n) y_k = \sum_{k=1}^{\infty} (a_k v_k + R_k) y_k.$$

(2) *In particular, if $a \in \{\eta(\Delta^\lambda)\}^\beta$ ($\eta = c$ or ℓ_∞); then we have the additional equalities:*

$$\sum_{k=1}^{\infty} (k + v_k) a_k = \lim_{n\to\infty} \sum_{k=1}^{n} (a_k v_k + R_k^n) = \sum_{k=1}^{\infty} (a_k v_k + R_k),$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} |a_k v_k + R_k^n| = \lim_{n\to\infty} \sum_{k=1}^{n} |a_k v_k + R_k| = \sum_{k=1}^{\infty} |a_k v_k + R_k|.$$

Proof. It is same as the proof of Corollary 3.6, part (1) is obtained from (3.8), and part (2) is immediate by (3.10) and noting that $\big|\,|a_k v_k + R_k^n| - |a_k v_k + R_k|\,\big| \le |R_k^n - R_k| = |R_{n+1}|$ for all $n \ge 1$ and every $k \le n$. □

Remark 3-6 It must be noted that Theorem 3.5 is reduced to Theorem 3.1 when $v = 0$. That is, the θ-duals of $\mu(\Delta)$, as in Theorem 3.1, can be obtained from θ-duals of $\mu(\Delta^\lambda)$, as in Theorem 3.5, with assuming that $v_k = 0$ for all k (Remark 3-5 is also reduced to Remark 3-2). Consequently, similar results of those in Corollaries 3.6 and 3.7 can be obtained for $\mu(\Delta)$ instead of $\mu(\Delta^\lambda)$ by taking $v_k = 0$ in these corollaries with $y_k = \Delta(x_k)$ instead of $y_k = \tilde{\Lambda}_k(x)$ for all k, where $x \in \mu(\Delta)$ in place of $x \in \mu(\Delta^\lambda)$. For instance, if $a \in \{\mu(\Delta)\}^\beta$; then for every $x \in \mu(\Delta)$, the relations given in Corollary 3.7 are satisfied with $v_k = 0$ and $y_k = \Delta(x_k)$ for all k. More precisely, the equalities in part (2) of Corollary 3.7 are respectively reduced to (2.13) and (2.14), where $a \in \{\eta(\Delta)\}^\beta$.

Corollary 3.8 *If $v \in \ell_\infty$; then $\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta$ for $\theta = \alpha$, β and γ, where $v = (\lambda_{k-1}/\Delta(\lambda_k))$ and μ is any of the spaces c_0, c or ℓ_∞.*

Proof. It is enough to show that if $v \in \ell_\infty$; then $\{\mu(\Delta)\}^\theta \subset \{a \in w : av \in \ell_1\}$ and so $\{\mu(\Delta^\lambda)\}^\theta = \{\mu(\Delta)\}^\theta$ by Theorem 3.4 (in such case, the condition $av \in \ell_1$ is redundant and so $\{\mu(\Delta^\lambda)\}^\theta$ is reduced to $\{\mu(\Delta)\}^\theta$). For this, suppose that $v \in \ell_\infty$. Then, for every $a \in \{\mu(\Delta)\}^\theta$, we have $a \in \ell_1$ (by Theorem 3.1) which implies that $av \in \ell_1$ (as $v \in \ell_\infty$). Thus, we deduce that $\{\mu(\Delta)\}^\theta \subset \{a \in w : av \in \ell_1\}$ when $v \in \ell_\infty$ and this completes the proof by Theorem 3.4. □

Remark 3-7 We may observe the following:

(1) The inclusions $c_0(\Delta) \subset c_0(\Delta^\lambda)$ and $\ell_\infty(\Delta) \subset \ell_\infty(\Delta^\lambda)$ imply both inclusions $\{c_0(\Delta^\lambda)\}^\theta \subset \{c_0(\Delta)\}^\theta$ and $\{\ell_\infty(\Delta^\lambda)\}^\theta \subset \{\ell_\infty(\Delta)\}^\theta$. This is suitable with Lemma 3.3, but Lemma 3.3 tells us also that $\{c(\Delta^\lambda)\}^\theta \subset \{c(\Delta)\}^\theta$ while the inclusion $c(\Delta) \subset c(\Delta^\lambda)$ need not be held. The justification can be understood in light of the equalities $\{c(\Delta)\}^\theta = \{\ell_\infty(\Delta)\}^\theta$ and $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta$. To see that, we have $\{c(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta^\lambda)\}^\theta \subset \{\ell_\infty(\Delta)\}^\theta = \{c(\Delta)\}^\theta$.

(2) It is known (see [14, Theorem 3.12]) that if $v \in \ell_\infty$; then $c_0(\Delta^\lambda) = c_0(\Delta)$ and $\ell_\infty(\Delta^\lambda) = \ell_\infty(\Delta)$. This implies that $\{c_0(\Delta^\lambda)\}^\theta = \{c_0(\Delta)\}^\theta$ and $\{\ell_\infty(\Delta^\lambda)\}^\theta = \{\ell_\infty(\Delta)\}^\theta$ which is suitable with Corollary 3.8, but Corollary 3.8 tells us also that $\{c(\Delta^\lambda)\}^\theta = \{c(\Delta)\}^\theta$ while the equality $c(\Delta^\lambda) = c(\Delta)$ need not be satisfied. Again, this can be justified by the same reason as in (1).

16

4 Certain Matrix Operators

In this last section, we have characterized some matrix classes and matrix operators related to the λ-difference spaces $\mu(\Delta^\lambda)$. We essentially deduced the necessary and sufficient conditions for an infinite matrix A to act on, into and between the spaces $\mu(\Delta^\lambda)$, where μ stands for any of the spaces c_0, c or ℓ_∞.

For any infinite matrix $A = [a_{nk}]$, we define the associated matrix $\tilde{A} = [\tilde{a}_{nk}]$ by

$$\tilde{a}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, a_{nk} + \sum\nolimits_{j=k}^{\infty} a_{nj} \qquad (n, k \geq 1), \tag{4.1}$$

where $A_n \in \ell_1$ for all $n \geq 1$. That is, the associated matrix $\tilde{A} = [\tilde{a}_{nk}]$ can be defined in terms of our notations given by (2.5) and (3.5) as follows:

$$\tilde{a}_{nk} = v_k a_{nk} + R_k(A_n) = v_k a_{nk} + R_{nk} \qquad (n, k \geq 1),$$

where $v = (v_k) = (\lambda_{k-1}/\Delta(\lambda_k))$, $A_n = (a_{nk})_{k=1}^{\infty}$ is the n-th row sequence in A $(n \geq 1)$ and $[R_{nk}]$ is an infinite matrix defined via A by $R_{nk} = R_k(A_n)$ for all $n, k \geq 1$, that is

$$v_k = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} \quad \text{and} \quad R_{nk} = \sum\nolimits_{j=k}^{\infty} a_{nj} \qquad (n, k \geq 1).$$

Further, assume the sequences $x, y \in w$ are connected by the relation $y = \tilde{\Lambda}(x)$. Then $x \in \mu(\Delta^\lambda) \iff y \in \mu$, and by using (3.6) with the same technique by which the relation (3.8) have been derived, we obtain that

$$\sum\nolimits_{k=1}^{m} a_{nk} x_k = \sum\nolimits_{k=1}^{m} \tilde{a}_{nk} y_k - R_{n,\,m+1}\, \sigma_m(y) \qquad (n, m \geq 1).$$

Moreover, if $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$; then it follows, by (2) of Theorem 3.5, that $A_n \in \ell_1$, $vA_n \in \ell_1$ and $R(A_n) = (R_{nk})_{k=1}^{\infty} \in \ell_1$ for all n. Also, we must have $\tilde{A}_n \in \ell_1$ and $\lim_{m\to\infty} R_{n,\,m+1} \sigma_m(y) = 0$ for all n and every $y \in \mu$ (see Corollary 3.2 and Remark 3-3), where $\tilde{A}_n = (\tilde{a}_{nk})_{k=1}^{\infty}$ is the n-th row sequence in the associated matrix $\tilde{A}$ for each $n \geq 1$, that is

$$\tilde{A}_n = vA_n + R(A_n) = \big(a_{nk} v_k + R_{nk}\big)_{k=1}^{\infty} \qquad (n \geq 1).$$

Thus, by going to the limits in both sides of above equality as $m \to \infty$, we get the following (see (1) of Corollary 3.7):

$$\sum\nolimits_{k=1}^{\infty} a_{nk} x_k = \sum\nolimits_{k=1}^{\infty} \tilde{a}_{nk} y_k \qquad (n \geq 1) \tag{4.2}$$

which means that $A_n(x) = \tilde{A}_n(y)$ for all n, and so $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^\lambda)$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$. This also means that $A(x) \in X$ for every $x \in \mu(\Delta^\lambda)$ if and only if $\tilde{A}(y) \in X$ for every $y \in \mu$, where X is any sequence space.

Lemma 4.1 *For any infinite matrix A, let $\tilde{A}$ be its associated matrix defined by* (4.1)*. Then, for each $n \geq 1$, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ if and only if $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$, where $\mu^\beta = \ell_1$ ($\mu = c_0$, c or ℓ_∞). Furthermore, if $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$; then $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^\lambda)$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$.*

17

Proof. Let $n \geq 1$. Then, by using Theorem 3.5 and Remark 3-3 with A_n instead of a, we deduce the following:

$$\begin{aligned} A_n \in \{\mu(\Delta^\lambda)\}^\beta &\Longleftrightarrow A_n \in \{\mu(\Delta)\}^\beta \text{ and } vA_n + R(A_n) \in \ell_1 \\ &\Longleftrightarrow A_n \in \{\mu(\Delta)\}^\beta \text{ and } \tilde{A}_n \in \mu^\beta, \end{aligned}$$

where $\mu^\beta = \ell_1$ (as $\mu = c_0$, c or ℓ_∞) and $\tilde{A}_n = vA_n + R(A_n)$ for all n. Further, if $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$; then it follows by (4.2) that $A(x) = \tilde{A}(y)$ for all $x \in \mu(\Delta^\lambda)$ and $y \in \mu$ which are connected by $y = \tilde{\Lambda}(x)$. This ends the proof. $\square$

Theorem 4.2 *For any sequence space X and every infinite matrix A, the following statements are equivalent to each others:*

(1) $A \in (\mu(\Delta^\lambda), X)$.

(2) $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ *for every* $n \geq 1$ *and* $\tilde{A}(y) \in X$ *for all* $y \in \mu$.

(3) $A_n \in \{\mu(\Delta)\}^\beta$ *for every* $n \geq 1$ *and* $\tilde{A} \in (\mu, X)$.

Proof. Suppose that (1) is satisfied, that is $A \in (\mu(\Delta^\lambda), X)$. Then $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$. Thus, for every $y \in \mu$, let $x = (x_k)$ be given by (3.6). Then $x \in \mu(\Delta^\lambda)$ such that $y = \tilde{\Lambda}(x)$ and so $\tilde{A}(y) \in X$ (as $A(x) = \tilde{A}(y)$ by Lemma 4.1) and since $y \in \mu$ was arbitrary; we find that $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $\tilde{A}(y) \in X$ for all $y \in \mu$ which is (2), that is (1) $\Longrightarrow$ (2). Further, assume that (2) is satisfied, that is $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $\tilde{A}(y) \in X$ for all $y \in \mu$. This, together with Lemma 4.1, implies that $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$ for every $n \geq 1$ as well as $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, we deduce that $A_n \in \{\mu(\Delta)\}^\beta$ for every $n \geq 1$ and $\tilde{A} \in (\mu, X)$ which is (3), that is (2) $\Longrightarrow$ (3). Finally, suppose that (3) is satisfied, that is $A_n \in \{\mu(\Delta)\}^\beta$ for every $n \geq 1$ and $\tilde{A} \in (\mu, X)$. This means that $A_n \in \{\mu(\Delta)\}^\beta$ and $\tilde{A}_n \in \mu^\beta$ for every $n \geq 1$ as well as $\tilde{A}(y) \in X$ for all $y \in \mu$. Hence, it follows by Lemma 4.1 that $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$. Besides, for every $x \in \mu(\Delta^\lambda)$, let $y = \tilde{\Lambda}(x)$. Then $y \in \mu$ and $A(x) = \tilde{A}(y)$ by (4.2) which implies that $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$. Therefore, we have $A_n \in \{\mu(\Delta^\lambda)\}^\beta$ for every $n \geq 1$ and $A(x) \in X$ for all $x \in \mu(\Delta^\lambda)$. This means that $A \in (\mu(\Delta^\lambda), X)$ which is (1), that is (3) $\Longrightarrow$ (1). This completes the proof. $\square$

Now, let's consider the following conditions in which we have used (4.1):

$$\sum_{k=1}^{\infty} |a_{nk}| \text{ converges for every } n \geq 1 \tag{4.3}$$

$$\left(k \sum_{j=k+1}^{\infty} a_{nj}\right)_{k=1}^{\infty} \in \ell_\infty \text{ for every } n \geq 1 \tag{4.4}$$

$$\left(k \sum_{j=k+1}^{\infty} a_{nj}\right)_{k=1}^{\infty} \in c_0 \text{ for every } n \geq 1 \tag{4.5}$$

$$\sum_{k=1}^{\infty} \left| \sum_{j=k}^{\infty} a_{nj} \right| \text{ converges for every } n \geq 1 \tag{4.6}$$

$$\sum_{k=1}^{\infty} |\tilde{a}_{nk}| \text{ converges for every } n \geq 1 \tag{4.7}$$

$$\sup_n \sum_{k=1}^{\infty} |\tilde{a}_{nk}| < \infty \tag{4.8}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = \tilde{a}_k \text{ exists for every } k \geq 1 \tag{4.9}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk} = \tilde{a} \text{ exists} \tag{4.10}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\tilde{a}_{nk} - \tilde{a}_k| = 0 \tag{4.11}$$

$$\lim_{n\to\infty} \tilde{a}_{nk} = 0 \text{ for every } k \geq 1 \tag{4.12}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk} = 0 \tag{4.13}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\tilde{a}_{nk}| = 0 \tag{4.14}$$

$$\sup_{K\in\mathcal{K}} \sum_{n=1}^{\infty} \Big| \sum_{k\in K} \tilde{a}_{nk} \Big|^p < \infty \text{ for } p \geq 1, \tag{4.15}$$

where $\mathcal{K}$ stands for the collection of all non-empty finite subsets of positive integers. Then, by using Theorems 3.1, 3.5 and Lemma 4.1, we find that

$$A_n \in \{c_0(\Delta)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (4.3), (4.4) \text{ and } (4.6) \text{ are satisfied,}$$

$$A_n \in \{\eta(\Delta)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (4.3), (4.5) \text{ and } (4.6) \text{ are satisfied,}$$

$$A_n \in \{c_0(\Delta^\lambda)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (4.3), (4.4), (4.6) \text{ and } (4.7) \text{ are satisfied,}$$

$$A_n \in \{\eta(\Delta^\lambda)\}^\beta \text{ for every } n \geq 1 \Longleftrightarrow (4.3), (4.5), (4.6) \text{ and } (4.7) \text{ are satisfied,}$$

where η stands for any of the spaces c or ℓ_∞ (also, it is obvious by Corollary 3.2 that conditions (4.4) and (4.5), in above equivalences, can be replaced by only one condition, viz: $\lim_{k\to\infty} \sigma_k(y) \sum_{j=k+1}^{\infty} a_{nj} = 0$ for all $y \in \mu$ and every $n \geq 1$). Therefore, by using Theorem 4.2 with help of those famous results of Stieglitz and Tietz [15] characterizing matrix operators between the classical sequence spaces, we can immediately deduce the following consequences characterizing matrix operators on the spaces $\mu(\Delta^\lambda)$:

Corollary 4.3 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), \ell_\infty)$ *if and only if* (4.3), (4.4), (4.6) *and* (4.8) *are satisfied.*

(2) $A \in (\eta(\Delta^\lambda), \ell_\infty)$ *if and only if* (4.3), (4.5), (4.6) *and* (4.8) *are satisfied.*

Proof. This result follows from Theorem 4.2, since $\tilde{A} \in (\mu, \ell_\infty) \Longleftrightarrow (4.8)$ held (note that: we have $(c(\Delta^\lambda), \ell_\infty) = (\ell_\infty(\Delta^\lambda), \ell_\infty)$ by part (2), where $\eta = c$ or ℓ_∞). □

Corollary 4.4 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), c)$ *if and only if* (4.3), (4.4), (4.6), (4.8) *and* (4.9) *are satisfied. Further, if* $A \in (c_0(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*, where* $y_k = \tilde{\Lambda}_k(x)$ *and* $\tilde{a}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ *for all* k*.*

(2) $A \in (c(\Delta^\lambda), c)$ *if and only if* (4.3), (4.5), (4.6), (4.8), (4.9) *and* (4.10) *are satisfied. Further, if* $A \in (c(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = L(\tilde{a} - \sum_{k=1}^{\infty} \tilde{a}_k) + \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*, where* $L = \lim_{k\to\infty} \tilde{\Lambda}_k(x)$ *and* $\tilde{a} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \tilde{a}_{nk}$*.*

(3) $A \in (\ell_\infty(\Delta^\lambda), c)$ *if and only if* (4.3), (4.5), (4.6), (4.8), (4.9) *and* (4.11) *are satisfied. Further, if* $A \in (\ell_\infty(\Delta^\lambda), c)$*; then* $\lim_{n\to\infty} A_n(x) = \sum_{k=1}^{\infty} \tilde{a}_k y_k$ *for every* $x \in \mu(\Delta^\lambda)$*.*

19

Proof. It is immediate by noting that: (1) $\tilde{A} \in (c_0, c) \Longleftrightarrow$ (4.8) and (4.9) are satisfied. (2) $\tilde{A} \in (c, c) \Longleftrightarrow$ (4.8), (4.9) and (4.10) held. (3) $\tilde{A} \in (\ell_\infty, c) \Longleftrightarrow$ (4.8), (4.9) and (4.11) are satisfied. $\square$

Corollary 4.5 *For an infinite matrix A, we have the following:*

(1) $A \in (c_0(\Delta^\lambda), c_0)$ *if and only if* (4.3), (4.4), (4.6), (4.8) *and* (4.12) *are satisfied.*

(2) $A \in (c(\Delta^\lambda), c_0)$ *if and only if* (4.3), (4.5), (4.6), (4.8), (4.12) *and* (4.13) *are satisfied.*

(3) $A \in (\ell_\infty(\Delta^\lambda), c_0)$ *if and only if* (4.3), (4.5), (4.6) *and* (4.14) *are satisfied.*

Proof. This follows by observing that: (1) $\tilde{A} \in (c_0, c_0) \Longleftrightarrow$ (4.8) and (4.12) satisfied. (2) $\tilde{A} \in (c, c_0) \Longleftrightarrow$ (4.8), (4.12) and (4.13) held. (3) $\tilde{A} \in (\ell_\infty, c_0) \Longleftrightarrow$ (4.14) held. $\square$

Corollary 4.6 *Let A be an infinite matrix. Then, for every real $p \geq 1$, we have:*

(1) $A \in (c_0(\Delta^\lambda), \ell_p)$ *if and only if* (4.3), (4.4), (4.6), (4.7) *and* (4.15) *are satisfied.*

(2) $A \in (\eta(\Delta^\lambda), \ell_p)$ *if and only if* (4.3), (4.5), (4.6), (4.7) *and* (4.15) *are satisfied.*

Proof. This follows by observing that: $\tilde{A} \in (\mu, \ell_p) \Longleftrightarrow$ (4.15) held (note that: we have $(c(\Delta^\lambda), \ell_p) = (\ell_\infty(\Delta^\lambda), \ell_p)$ by part (2), where $p \geq 1$ and $\eta = c$ or ℓ_∞). $\square$

Further, in the light of Remark 3-6, it must be noted that Corollaries 4.3, 4.4, 4.5 and 4.6 can be reduced, with assumption $v = 0$, to characterize matrix operators on the difference spaces $\mu(\Delta)$ as follows:

Remark 4-1 The necessary and sufficient conditions for an infinite matrix A in order to belong to any of the classes $(\mu(\Delta), \ell_\infty)$, $(\mu(\Delta), c)$, $(\mu(\Delta), c_0)$ or $(\mu(\Delta), \ell_p)$ are those conditions given respectively in Corollary 4.3, 4.4, 4.5 or 4.6 by removing condition (4.6) and taking $\tilde{a}_{nk} = R_{nk} = \sum_{j=k}^\infty a_{nj}$ for all $n, k \geq 1$, where $p \geq 1$.

Furthermore, we have the following useful result (cf. [8, 10, 13]):

Lemma 4.7 *Let X and Y be sequence spaces, A an infinite matrix and T a triangle. Then $A \in (X, Y_T)$ if and only if $TA \in (X, Y)$.*

It is obvious that Lemma 4.7 can be used to characterize matrix operators acting from $\mu(\Delta^\lambda)$ into matrix domains of triangles. For instance, we have $cs_0 = (c_0)_\sigma$, $cs = (c)_\sigma$, $bs = (\ell_\infty)_\sigma$, $c_0(\Delta) = (c_0)_\Delta$, $c(\Delta) = (c)_\Delta$, $\ell_\infty(\Delta) = (\ell_\infty)_\Delta$ and $bv_p = (\ell_p)_\Delta$ for $p \geq 1$. Therefore, we conclude the following consequences:

Corollary 4.8 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\tilde{b}_{nk}]$ by*

$$b_{nk} = a_{nk} - a_{n-1,k} \quad \text{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}} b_{nk} + \sum_{j=k}^\infty b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^\infty b_{nk}$ converge for all $n \geq 1$. Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\mu(\Delta^\lambda), \ell_\infty(\Delta))$, $(\mu(\Delta^\lambda), c(\Delta))$, $(\mu(\Delta^\lambda), c_0(\Delta))$ or $(\mu(\Delta^\lambda), bv_p)$ are those conditions given respectively in Corollary 4.3, 4.4, 4.5 or 4.6 provided that the entries a_{nk} and $\tilde{a}_{nk}$ are respectively replaced by b_{nk} and $\tilde{b}_{nk}$ for all $n, k \geq 1$, where $p \geq 1$.

20

Corollary 4.9 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\tilde{b}_{nk}]$ by*

$$b_{nk} = \sum\nolimits_{j=1}^{n} a_{jk} \quad \text{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum\nolimits_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where the series $\sum_{k=1}^{\infty} b_{nk}$ converge for all $n \geq 1$. Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\mu(\Delta^{\lambda}), bs)$, $(\mu(\Delta^{\lambda}), cs)$ or $(\mu(\Delta^{\lambda}), cs_0)$ are those conditions given respectively in Corollary 4.3, 4.4 or 4.5 provided that the entries a_{nk} and $\tilde{a}_{nk}$ are respectively replaced by b_{nk} and $\tilde{b}_{nk}$ for all $n, k \geq 1$.

Moreover, by means of Lemma 4.7, we can obtain the characterization for matrix operators acting from the classical sequence spaces into $\mu(\Delta^{\lambda})$ as follows:

Corollary 4.10 *Let A be an infinite matrix and define the matrix $B = [b_{nk}]$ by*

$$b_{nk} = \frac{\Delta(\lambda_n)}{\lambda_n}\, a_{nk} + \Delta\Big(\frac{1}{\lambda_n}\Big) \sum\nolimits_{j=0}^{n-1} \Delta(\lambda_j)\, a_{jk} \qquad (n, k \geq 1).$$

Then A belongs to any one of the classes $(c_0, \mu(\Delta^{\lambda}))$, $(c, \mu(\Delta^{\lambda}))$, $(\ell_{\infty}, \mu(\Delta^{\lambda}))$ or $(\ell_p, \mu(\Delta^{\lambda}))$ if and only if B belongs to the respective one of the classes (c_0, μ), (c, μ), (ℓ_{∞}, μ) or (ℓ_p, μ), where $p \geq 1$ and μ stands for any of the spaces c_0, c or ℓ_{∞}.

Remark 4-2 We refer the reader to [15] for characterization of matrix classes (c_0, μ), (c, μ), (ℓ_{∞}, μ) or (ℓ_p, μ) mentioned in Corollary 4.10, where $p \geq 1$. Also, it is clear that Corollary 4.10 can be generalized to characterize matrix operators acting from an arbitrary sequence space X into $\mu(\Delta^{\lambda})$, where $A \in (X, \mu(\Delta^{\lambda})) \iff B \in (X, \mu)$.

Finally, we conclude our work with the following corollary characterizing matrix operators between the spaces $\mu(\Delta^{\lambda})$. For this, let $\lambda' = (\lambda'_k)$ be a strictly increasing sequence of positive reals (λ and λ' need not be equal). Then $\mu(\Delta^{\lambda'}) = (\mu)_{\Lambda'}$, where Λ' is the triangle defined by (2.3) with λ' instead of λ (see (2.4)).

Corollary 4.11 *Let A be an infinite matrix and define the matrices $[b_{nk}]$ and $[\tilde{b}_{nk}]$ by*

$$b_{nk} = \frac{\Delta(\lambda'_n)}{\lambda'_n}\, a_{nk} + \Delta\Big(\frac{1}{\lambda'_n}\Big) \sum\nolimits_{j=0}^{n-1} \Delta(\lambda'_j)\, a_{jk} \qquad (n, k \geq 1),$$

$$\tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum\nolimits_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1),$$

where $\sum_{k=1}^{\infty} b_{nk}$ converge for all $n \geq 1$. Then, the necessary and sufficient conditions in order that A belongs to any one of the classes $(\mu(\Delta^{\lambda}), \ell_{\infty}(\Delta^{\lambda'}))$, $(\mu(\Delta^{\lambda}), c(\Delta^{\lambda'}))$ or $(\mu(\Delta^{\lambda}), c_0(\Delta^{\lambda'}))$ are those conditions given respectively in Corollary 4.3, 4.4 or 4.5 provided that a_{nk} and $\tilde{a}_{nk}$ are respectively replaced by b_{nk} and $\tilde{b}_{nk}$ for all $n, k \geq 1$.

Remark 4-3 By following the same technique used in Corollary 4.11, we can deduce the characterization of matrix operators acting from $\mu(\Delta^{\lambda})$ into any of the λ'-sequence spaces $c_0^{\lambda'}$, $c^{\lambda'}$, $\ell_{\infty}^{\lambda'}$ or $\ell_p^{\lambda'}$ $(p \geq 1)$ defined in [11] and [12], where (see (2.2))

$$b_{nk} = \frac{1}{\lambda'_n} \sum\nolimits_{j=1}^{n} \Delta(\lambda'_j)\, a_{jk} \quad \text{and} \quad \tilde{b}_{nk} = \frac{\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\, b_{nk} + \sum\nolimits_{j=k}^{\infty} b_{nj} \qquad (n, k \geq 1).$$

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