# Homotopy Analysis Method for Large-Amplitude Nonlinear Vibration of Flat Plates

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***Abstract :*** *In this study, the homotopy analysis method is employed to analyze nonlinear vibration of a flat plate by simply supports. The governing equation of vibration is derived based on large-amplitude assumption. Also, the effects of shear deformation and rotary inertia of the cross section of the plate is considered. Because of these factors, the cubic nonlinear terms are created in the characteristic equation of vibration. To transform partial differential equations into ordinary differential equation, Galerkin method is used. After solving of this equation, the analytical relationship for natural frequency of vibration is obtained. With employing of this relationship, the effect of design parameters on vibration frequency is investigated. In order to accurately assess and examine the precision of analyses, the results with the numerical solution by Runge- Kutta method fourth order are compared. The results show that in this case, the respond of analytical method has a good correspondence with a numerical method. According of results, increasing of thickness ratio to width of plate and Poisson's ratio caused the natural nonlinear frequency increases. Furthermore, the obtained results show that, increasing the factor term of rotary inertia effect caused frequency increases, where increasing the factor term of shear deformation caused frequency decreases.*

***Keywords:*** *Nonlinear vibration, Homotopy analysis method, Shear deformation, Flat plates.*

## I. INTRODUCTION

Plate members are one of the efficient structural members that are using in a various engineering applications, such as: wing of aeroplate structures, large bridges, and giant spacecrafts. Regarding to these extensive applications, the possibility of creating the vibrations larger than plate thickness that are known Large-amplitude Vibration is existed. Flat plates with cantilever boundary condition along the fixed edges, show the hardening behavior when they are subjected in a large-amplitude of nonlinear vibration [1].

In such cases, the creating sources of nonlinear terms in equations governing vibration of plate, is more important. These nonlinear sources are classified geometry, inertia or material. The nonlinear terms of geometry are existed in causing of large-amplitude vibrations. The nonlinear equation that refer to stress and strain relationship, can created the nonlinear of material. Also, concentrated mass and asymmetrically distributed caused the nonlinear created inertia [2]. Major investigations on nonlinear vibration of plates by Amabili, Sathyamoorthy and Chia was presented [3-5]. In this context, Chia and Herrmann have simple investigations for rectangular plates by simply supported to obtain the time response of linear and free vibrations [6]. Also, in some researches, method of R function have been used in order study nonlinear vibration of plates and thin shells with different forms and various boundary condition [7-9]. Many investigations study large-amplitude vibration of rectangular plates with various boundary conditions by using of varied technics. Chang and Chian employing of Galerkin and elliptical integral methods for study free nonlinear vibration of orthotropic form of rectangular plates in first and second modes [10]. Large-amplitude vibration of rectangular plates with various combination of simply and cantilever supports were studied by Saha [11].

Eslami and co-workers employed multiple time scale and Galerkin methods in investigating of nonlinear forced vibration of orthotropic plates for simply movable supports [12, 13]. The vibration of flexible plates with geometry nonlinear terms investigated by Ganapathi based on finite element method [14]. Amabili investigated the response time of nonlinear vibration plates considering of a nonlinear geometry term by analytical method; afterward, the obtained results compared with experimental results [15-17]. In the most aforementioned literatures, has been used traditional perturbation methods in order analytical solving. These methods have ability to solve the problems with nonlinear terms of small factors [18].

In recent years, in order to solve the complex nonlinear problems, strong and powerful methods have been developed. Homotopy method is one of these methods which are using for solve of various nonlinear problems.

Pirbodaghi and co-workers by using this method investigated the nonlinear behavior of beam under axial loads. Hence, an appropriate term for expression nonlinear frequency is obtained [19]. Also, Rafieipor and co-workers employed Homotopy method for analysis of free vibrations in a nonlinear elastic beam that were under of thermal and mechanical loads [20].

In the present study, in deriving of governing equations on vibration of a rectangular plate, the nonlinear term of large-amplitude vibrations are considered. Furthermore, with respect to effects of shear deformation and rotary inertia of the cross section of the plate, the other cubic nonlinear terms are created in equation. To convert partial differential equations to ordinary differential equation, Galerkin method is employed. In order to homotopy method is employed. By using of this method, analytical relation for natural frequency of the nonlinear vibration plate is obtained. With employing of this method, the effects of design parameters could be evaluated, like as: thickness ratio to dimension of plate and Poisson's ratio to vibration frequency of flat plates. Also, with this expression the effects of nonlinear terms on vibration of system is examined.

## II. Formulation

The governing equation of transvers vibration of a flat plate, which have large-amplitude is in form of equation (1). In this equation, the effects of shear deformation and rotary inertia of plate sections are considered, as can be seen in figure 1[21,22].

$$\begin{aligned}&\left[\frac{D^2(1-\upsilon)}{2}\nabla^4-\frac{D\rho h^3(3-\upsilon)}{24}\nabla^2\frac{\partial^2}{\partial t^2}-\frac{5DEh^3(3-\upsilon)}{24(1+\upsilon)}\nabla^2+\frac{25E^2h^2}{144(1+\upsilon)^2}+\frac{5\rho Eh^4}{72(1+\upsilon)}\frac{\partial^2}{\partial t^2}\right]\times\\&\left[\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}-2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2\varphi}{\partial y\partial x}+\frac{5Eh}{12(1+\upsilon)}\nabla^2 w-\rho h\ddot{w}\right]+\\&\frac{5Eh}{12(1+\upsilon)}\left[\frac{5DEh(1-\upsilon)}{24(1+\upsilon)}\nabla^4-\frac{25E^2h^2}{144(1+\upsilon)^2}\nabla^2-\frac{25\rho Eh^4}{144(1+\upsilon)}\frac{\partial^2}{\partial t^2}\right]\times w=0\end{aligned}\tag{1}$$

$$\nabla^4\varphi=Eh_0\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial^2 x}\frac{\partial^2 w}{\partial^2 y}\right]$$

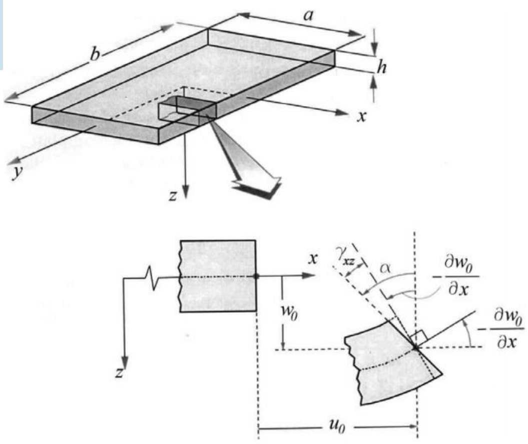


**Fig. 1.** Rectangular isotropic plate and its displacement after deflection.

In equation (2), u is poisson's ratio. To transform partial differential equation (1) into ordinary differential equation, Galerkin method is used [19, 20]. For this purpose, the solution of (1) is considered as follows.

$$D=\frac{Eh_0^{\ 3}}{12\left(1-\upsilon^2\right)}\tag{2}$$

In equation (2), u is poisson's ratio. To transform partial differential equation (1) into ordinary differential equation, Galerkin method is used [19, 20]. For this purpose, the solution of (1) is considered as follows.

$$w(x,y,t)=\psi(x,y)f(t)\tag{3}$$

In continue, in order to use of this method, y(x,y) is assumed such as linear mode of plate. F(t) is amplitude of plate vibration and is dependent of time. Dimensions of plate in directions of x and y are considered to be equal to each other. The first linear mode of plate vibration which have simply movable supports is form as.

$$\psi_{11}(x,y)=h_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right) \tag{4}$$

In aforementioned equation, a is dimension of plate in direction of x and y. in continue, to transform partial differential equation (1) into ordinary differential equation, at first, the equation (4) substituting in equation (3). Then resulting equation, is placed in equation (1). In present problem, the form of first plate mode is considered as weight function to apply the Galerkin method. By using of this method and doing simply mathematical oprations, the ordinary differential equation is achieved, finally.

$$\ddot{f}(t)+b_1 f(t)+b_2 f(t)^3+b_3 f(t)^2\ddot{f}(t)+b_4 f(t)\dot{f}(t)^2=0 \tag{5}$$

The resulting equation is called the characteristic equation of the vibration plate, which is in the first mode. In this equation, $b_1$, $b_2$, $b_3$ and $b_4$ are functions of poisson's ratio u and ratio of $r=h_0/a$ where are in the appendix. The cubic nonlinear term ( $b_2f^3$ ) is related to vibration plate of large-amplitude. The other nonlinear terms in this equation, considering of shear deformation effects and rotary inertia of plate sections is created, which is a kind of nonlinear inertia [21]. In continue, the solution of equation (5) with regarding the initial condition have been considered.

$$f(0)=a_0, \quad \dot{f}(0)=0.$$

These condition indicates, that plate in the first form mode is effected of a particular amplitude, then, without apply the initial speed, is released. In this case, plate starts to vibration freely.

## III. HOMOTOPY ANALYSIS METHOD

## 3 HOMOTOPY ANALYSIS METHOD

### *3.1 Basic idea*

Homotopy is analytical method for solving the nonlinear differential equations [24-26]. This method, transform the nonlinear differential equation into infinite linear differential equation [24]. As Hmotopy parameter p, varies from zero to one, the problem of a preliminary estimation will tend to the exact solution. To illustrate the basic idea of homotopy method, a nonlinear differential equation can be considered as follows:

$$N[f(t)]=0, \quad f(0)=a_0, \quad \dot{f}(0)=0 \tag{6}$$

N nonlinear differential operator and f (t) is a function of the variable t that is assumed to be unknown. For expression (6), homotopy phrase is defined as follows [27]:

$$\bar{H}(\varphi,\mathrm{p},\mathrm{h},H(t))=(1-p)L[\varphi(t,p)-f_0(t)]-phH(t)\,N[\varphi(t,\mathrm{p})] \tag{7}$$

In equation (7), φ is a function of t and p. Also, h and H(t) are arbitrary non-zero parameter and function, respectively. Arbitrary parameter and function, set the convergence of considering solution [27]. L shows a linear and arbitrary operator. When P tend from zero to 1, φ(t,p) of initial estimate for the solution of the differential equation will change to exact solution. In the other hand, φ(t,0)=f0(t) that is solution of homotopyexpression $H(\varphi,p,h,H(t))\big|_{p=0}=0$, changes to homotopy expression $H(\varphi,p,h,H(t))\big|_{p=1}=0$. Apply H(φ,p,h,H(t))=0, provide deformation of zero-order homotopy as follows:

$$(1-p)L[\varphi(t,p)-f_0(t)]=phH(t)\mathrm{N}[\varphi(t,p)] \tag{8}$$

Equation (8) with initial condition is in form of fallows:

$$\varphi(0,p)=a_0\ ,\ \frac{d\varphi(0,p)}{dt}=0 \tag{9}$$

Functions φ(t, p) and ω(p) using the Taylor theory, extended governing equation expanded and can be formulated as a power series p:

$$\varphi(t,p)=\varphi(t,0)+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\varphi(t,p)}{\partial p^m}\bigg|_{p=0}p^m=f_0(t)+\sum_{m=1}^{\infty}f_m(t)p^m \tag{10}$$

$$\omega(p)=\omega_0+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\omega(p)}{\partial p^m}\bigg|_{p=0}p^m=\omega_0+\sum_{m=1}^{\infty}\omega_m p^m \tag{11}$$

$\omega_m(p)$ and $f_m(t)$ is called the deformation degree of m. To calculate the equation of first-order deformation, can be derived from equation (8) with respect to p. Then with putting zero in value of p, the equation of first-order deformation is obtained as:

$$L[f_1(t)]=hH(t)N[f_0(t),\omega_0]\big|_{q=0} \tag{12}$$

By using of equation (12), another approximation of solution for nonlinear differential equation could be obtained. The initial condition of equation (12) is the fallowing as:

$$f_1(0)=0\ ,\ \frac{df_1}{dt}(t)=0 \tag{13}$$

Higher-order approximations of solution can be calculated by deformation equation in order of m(m>1). Which their homotopy form expressed as:

$$L[f_m(t)-f_{m-1}(t)]=hH(t)R_m(f_{m-1},\vec{\omega}_{m-1}) \tag{14}$$

$f_{m-1}$, $\omega_{m-1}$ and $R_m(f_{m-1},\omega_{m-1})$ are calculated as fallows:

$$R_m(\vec{f}_{m-1},\vec{\omega}_{m-1})=\frac{1}{(m-1)!}\frac{d^{m-1}N[\varphi(t,p),\omega(p)]}{dp^{m-1}}\bigg|_{p=0} \tag{15}$$

$$\vec{q}_{m-1}=\{f_0,f_1,f_2,\ldots\ldots,f_{m-1}\}$$
$$\vec{\omega}_{m-1}=\{\omega_0,\omega_1,\omega_2,\ldots\ldots,\omega_{m-1}\} \tag{16}$$

The equation of higher-order approximation have initial condition as:

$$f_m(0)=\frac{df_m}{dt}(0)=0 \tag{17}$$

***3.2 Application of Homotopy method in present problem***

In order to use of homotopy method, for solving the characteristic equation of the plate vibration , the variable of $\tau=\omega t$ is defined. The new variable is applied in equation (5), and then rewritten as follows:

$$\omega^2\ddot{f}+b_1f+b_2f^3+b_3\omega^2f^2f''+b_4\omega^2f\,f'^2=0$$
$$\begin{cases}f(0)=a_0\\ f'(0)=0\end{cases} \qquad 18)$$

In equation (18), the sign of prime represent derivation with respect to variable of t. Nonlinear operator that introduced in homotopy method to equation (18), formulated as:

$$N\left[\varphi(\tau,p)\right]=\Omega^2(p)\frac{\partial^2\varphi(\tau,p)}{\partial\tau^2}+b_1\varphi(\tau,p)+b_3\varphi^3(\tau,p)+b_3\Omega^2(p)\varphi^2(\tau,p)\frac{\partial\varphi(\tau,p)}{\partial\tau^2} \qquad 19)$$
$$+b_4\Omega^2(p)\varphi(\tau,p)\left(\frac{\partial\varphi(\tau,p)}{\partial\tau}\right)^2$$

Due to the solution, an initial estimation for the solution of equation (18) can be considered. It should be noted, that this estimation, satisfying the initial conditions of problem, so

$$f_0(\tau)=a_0\cos(\tau) \qquad 20)$$

As can be seen, the equation (20) satisfy the initial condition of equation (18). In continue, arbitrary linear operator for this problem is chosen as follows:

$$L\left[\varphi(\tau,p)\right]=\omega_0^2\left[\frac{\partial\varphi(\tau,p)}{\partial\tau^2}+\varphi(\tau,p)\right] \qquad 21)$$

$$\omega_0^2\left[\frac{\partial f_1(\tau)}{\partial\tau^2}+f_1(\tau)\right]=-a_0\,h\omega_0^2\cos\tau+b_1\,a_0\,h\cos\tau+b_3\,a_0^3\,h\cos^3\tau-a_0^3hb_3\omega_0^2\cos^3\tau \tag{22}$$

$$+a_0^3\,hb_4\omega_0^2\ \cos\tau\sin^2\tau$$

As expressed above, in homotopy method, the final solution of the nonlinear equation obtained of algebraic summation in initial estimation and higher order approximations. Therefor, it must avoided of formation big time terms and unstable times such as tcos(t). In order to prevent emergence of big terms in time response, the coefficient of the term cos( $\tau$ ) should set to zero. As a result:

$$\omega_0=\frac{\sqrt{\left(-b_4a_0^2+3a_0^2b_3+4\right)\left(4b_1+3a_0^2b_2\right)}}{-b_4a_0^2+3a_0^2+4} \tag{23}$$

Analytical expression (23) is the first approximation for nonlinear natural frequency of plate vibration. Then response of equation (22), by considering of initial condition of equation (17) is obtained as;

$$f_1(\tau)=A_1(\cos3\tau-\cos\tau)$$

$$A_1=\frac{a_0^3\left(\left(b_3+b_4\right)\omega_0^2-b_2\right)h}{32\omega_0^2} \tag{24}$$

First approximation to describe the time-dependent amplitude of plate vibration, are calculated using equation (10) in the algebraic summation of equations (19) and (24), as result:

$$f(\tau)=a_0\cos(\tau)+A_1(\cos3\tau-\cos\tau)$$

In this way, the higher approximation of solution for nonlinear natural frequency and time-dependent amplitude of vibrations can be obtained.

## *4 RESULTS AND DISCUSSION*

In this section, the accuracy of the analysis that performed in this study, will be investigated, firstly. For this purpose, the results are compared numerically with Runge-Kutta method fourth order. In this regard, the equation (5), has been solved by Maple software with applying the initial condition. As can be seen in Fig.2, data from analytical method are entirely consistent with numerical results, that represents a very high accuracy of analysis.

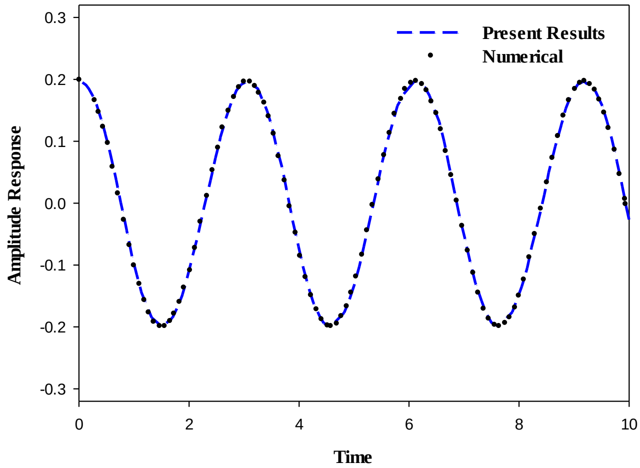


**Fig. 2**. Comparing of numerical method and analytical method.

To evaluate the importance of nonlinear terms in analysis of problem, the effects of these terms on response of time-dependent amplitude are investigated in figures 3 and 4. As shown in these figures, with regardless of some nonlinear terms, period of vibration decreases or increases. As can be seen, the elimination of nonlinear term from rotary inertia, lead to decreases of vibration period.

However, the elimination of nonlinear terms of large-amplitude vibration, and also, shear deformation of section caused the period of vibrations increases. Of course, as indicated in Figure 4, the effect of nonlinear terms represents the vibration of large-amplitude is more than.
The effect of initial condition (amplitude of initial excitation) on nonlinear natural frequency of system are shown in figures of 5 and 6 in two cases: different poisson's ratio and different of r ratio. In both figures, increasing of poisson's and r ratios causes the nonlinear natural frequency of system increased. In figures of 7, 8, and 9 the different effects of equation factor on nonlinear natural frequency is investigated. It is observed, the increasing of nonlinear factor-term from inertia, causes nonlinear natural frequency of plate decreased. Although, increasing of nonlinear factor-terms from stiffness and shear deformation causes the frequency increased.

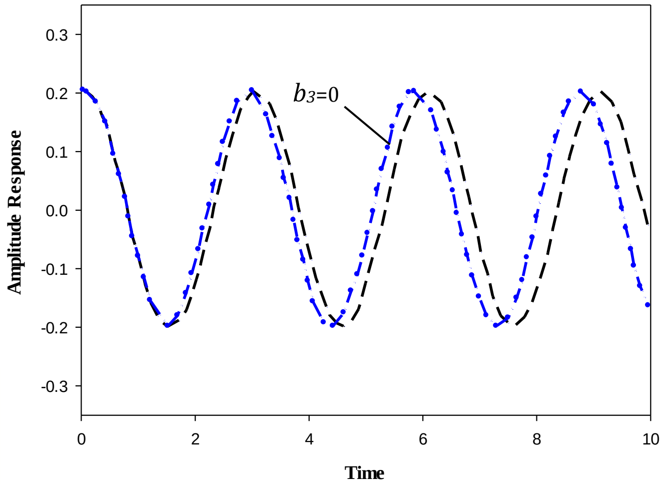


**Fig. 3**. The effect of elimination the nonlinear term with inertia $b_3$.

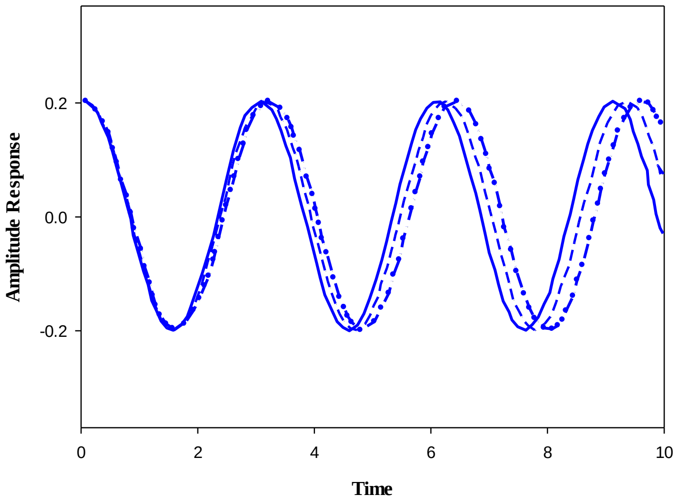


**Fig. 4**. The effect of elimination the nonlinear term with stiffness $b_2$ and shear deformation $b_4$.

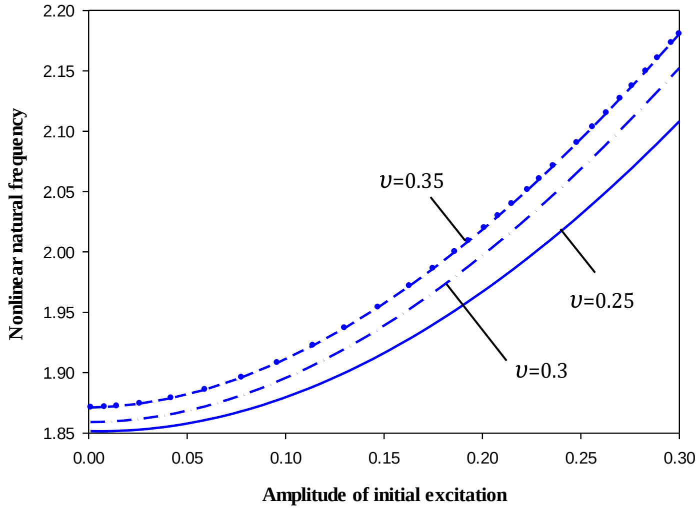


**Fig. 5**. Effect of poissons ratio on nonlinear frequency.

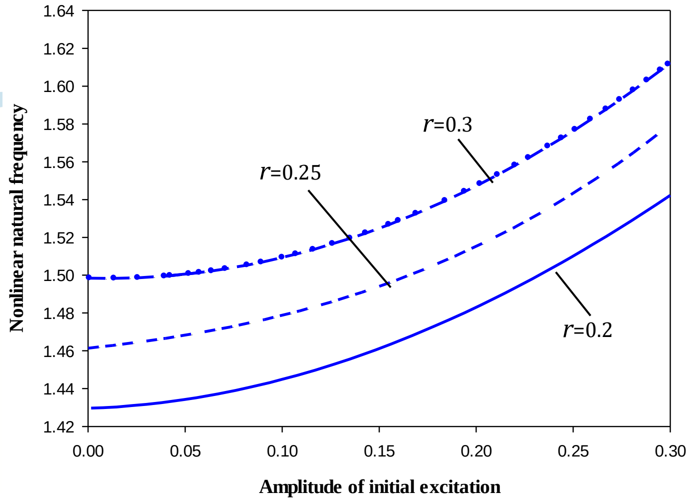


**Fig. 6**. Effect of r ratio on nonlinear natural frequency.

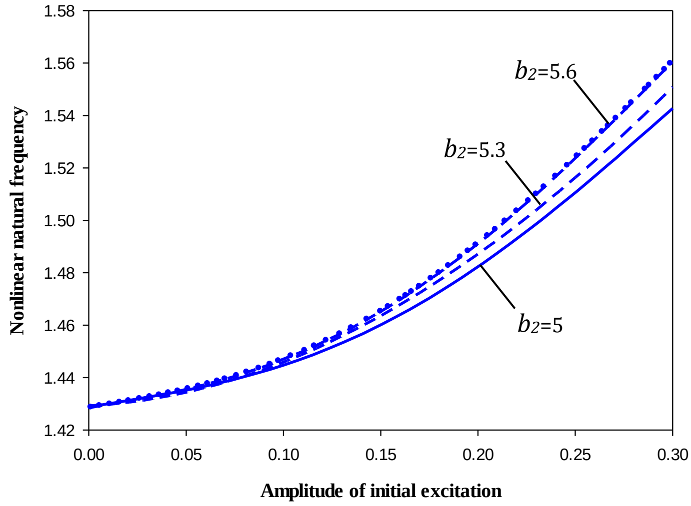


**Fig. 7**. Effect of nonlinear factor $b_2$ on nonlinear frequency.

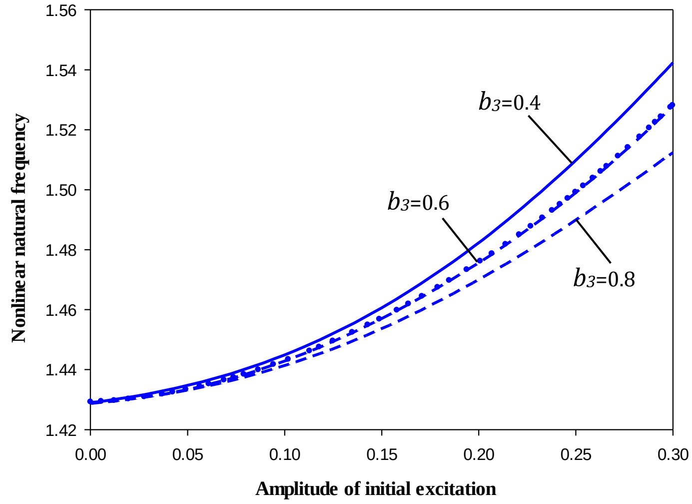


**Fig. 8**. Effect of nonlinear factor $b_3$ on nonlinear frequency.

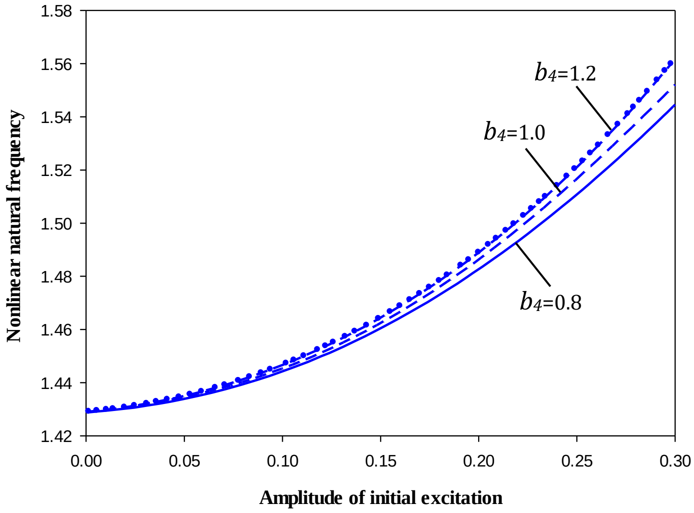


**Fig. 9**. Effect of nonlinear factor $b_4$ on nonlinear frequency**.**

## 5 Conclusion

In the present study, considering the rotary inertia effects and shear deformation, large-amplitude vibration and nonlinearity of a flat plate with simply supports has been studied. To analysis and arrive to analytical expression, was employing of homotopy method in order to represented the vibration of nonlinear natural frequency. The comparison of numerical solution and analytical solution, indicating high accuracy of calculated that performed. The results of parametric investigations show that increasing of poisson's ratio and relation of width to thickness of plate causes nonlinear natural frequency increased. Also, accroding to results the effects of nonlinear factors on large-amplitude vibration in comparison small-amplitude vibration is more than. Therefore, in order to accurate prediction of behavior of plate vibration, nonlinear terms should be considered.

## 6 APPENDICES

$$b_1 = \frac{10\pi^4 r^2(\pi^2 r^2 + 5)}{(2\pi^4 r^4 + 115\pi^2 r^2 + 60\pi^2 r^2 \vartheta - 55\pi^2 r^2 \vartheta^2 + 150 - 150\vartheta^2)} \qquad (1)$$

$$b_2 = \frac{\begin{pmatrix} -5\pi^2 r^2 \upsilon^3 - 25\upsilon^3 + 2\pi^4 r^4 \upsilon^2 + 15\pi^2 r^2 \upsilon^2 - \\ 25\upsilon^2 - 5\pi^2 r^2 \upsilon + 25\upsilon + 2\pi^4 r^4 + 15\pi^2 r^2 + 25 \end{pmatrix}}{\begin{pmatrix} 2\pi^4 r^4 + 115\pi^2 r^2 + 60\pi^2 r^2 \upsilon \\ -55\pi^2 r^2 \upsilon^2 + 150 - 150\upsilon^2 \end{pmatrix}} \frac{(\frac{3}{4})(1+\upsilon)\pi^4 r^2 (-3+\upsilon)}{(1+\upsilon^2)(-1+\upsilon)} \tag{2}$$

$$b_3 = -\frac{\left(-3\pi^2 r^2 + \pi^2 r^2 \upsilon - 10 + 10\upsilon\right)}{\begin{pmatrix} 2\pi^4 r^4 + 115\pi^2 r^2 + 60\pi^2 r^2 \upsilon - \\ 55\pi^2 r^2 \upsilon^2 + 150 - 150\upsilon^2 \end{pmatrix}} \frac{9\pi^4 r^4 (1+\upsilon)^2 (-3+\upsilon)}{4(-1+\upsilon)} \tag{3}$$

$$b_4 = -\frac{\left(-3\pi^2 r^2 + \pi^2 r^2 \upsilon - 10 + 10\upsilon\right)}{\begin{pmatrix} 2\pi^4 r^4 + 115\pi^2 r^2 + 60\pi^2 r^2 \upsilon - \\ 55\pi^2 r^2 \upsilon^2 + 150 - 150\upsilon^2 \end{pmatrix}} \frac{9\pi^4 r^4 (-3+\upsilon)(1+\upsilon)^2}{2(-1+\upsilon)} \tag{4}$$

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